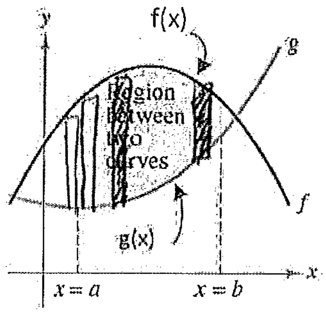
Key

## AP Calculus Ch. 7.1 – Area Between Two Curves




**Vertical Orientation: (vertical rectangles between graphs)**

Right bound

$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

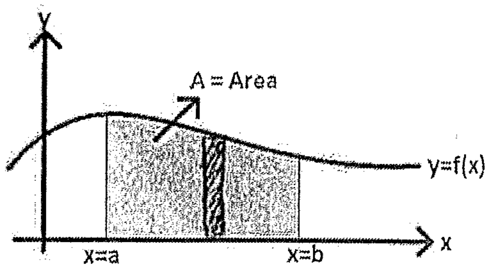
Left bound

Expressions in terms of x

(Equations in the form of " y = ____ ")

**Example 1:** Area = $\int_a^b f(x) - g(x)\, dx$

**Example 2:**



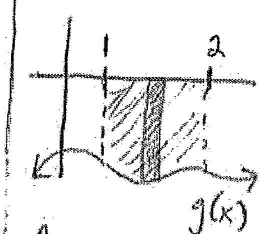

Top graph

bottom graph

$\int_a^b f(x) - 0\, dx$

Area = $\int_a^b f(x)\, dx$

Ex. 2b




Area = $\int_1^2 0 - g(x)\, dx$

$\boxed{\int_1^2 -g(x)\, dx}$

**Example 3:** Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$

**Steps:**

i) **Find bounds**: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for x).

ii) Identify the **top and bottom** function

iii) Apply the Integral **Area Formula.**

* set equations equal:

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3,\ x = -1$

Top

bottom

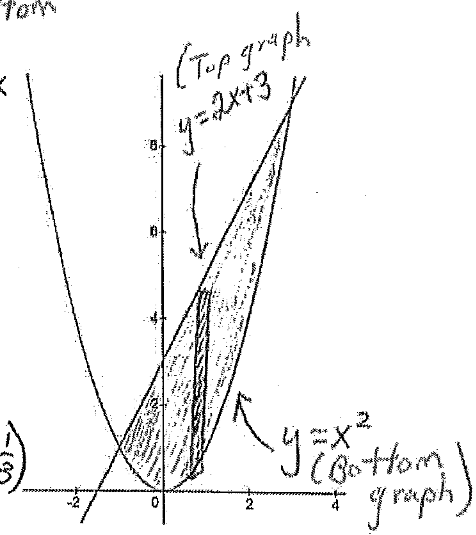
$$Area = \int_{-1}^{3} (2x+3) - (x^2)\, dx$$

$$\int 2x + 3 - x^2\, dx$$

$$\left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

$$3^2 + 3(3) - \frac{3^3}{3} - \left((-1)^2 + 3(-1) + \frac{1}{3}\right)$$

$$\boxed{Area = \frac{32}{3}}$$



**Horizontal Orientation: (horizontal rectangles between graphs)**

Upper bound

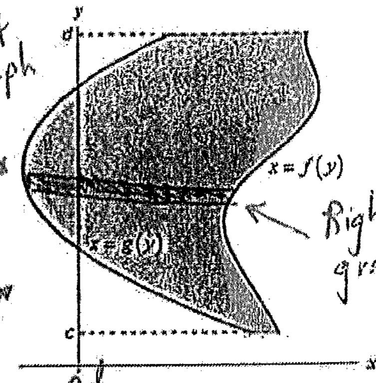
$$Area = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)\, dy$$

Lower bound

Expressions in terms of y

(Equations in the form of "x = ____")

left graph




Right graph

**Example 3:** Area = $\int_c^d f(y) - g(y)\, dy$

**Example 4**: Find area of the region bounded by the equations on right:

**Steps:**

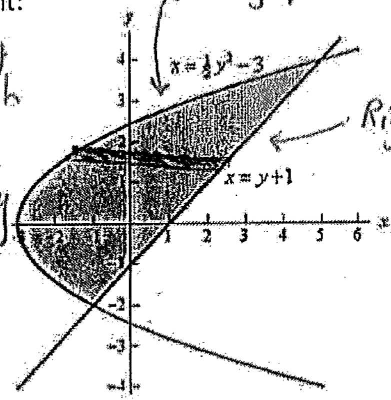
**i) Find bounds:** Find the point of intersection between the 2 graphs

(by setting equations equal, & solving for y).

**ii)** Identify the **right and left** function

**iii)** Apply the Integral **Area Formula**

Left graph




Right graph

* find bounds (intersection)

$\frac{1}{2}y^2 - 3 = y + 1$

$2\left(\frac{1}{2}y^2 - y - 4 = 0\right)$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = 4,\ y = -2$

Right graph Left graph

$Area = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$

$\int y + 1 - \frac{1}{2}y^2 + 3\, dy$

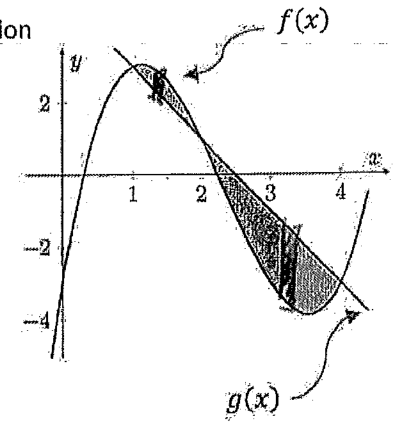
$\int y - \frac{1}{2}y^2 + 4\, dy$

$\frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^3}{3} + 4y \Big]_{-2}^{4} = \frac{4^2}{2} - \frac{4^3}{6} + 4(4) - \left(\frac{-2^2}{2} - \frac{(-2)^3}{6} - 8\right) = 18$

$\boxed{Area = 18}$

**Example 5:** Represent the area of shaded region to the right using integral notation

$$\boxed{Area = \int_1^2 f(x) - g(x)\, dx + \int_2^4 g(x) - f(x)\, dx}$$

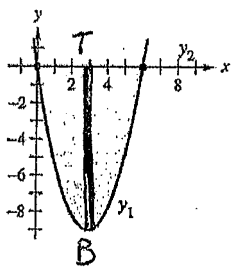


Key

$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

**Writing a Definite Integral** In Exercises 1–6, set up the definite integral that gives the area of the region.

1. $y_1 = x^2 - 6x$
   $y_2 = 0$




$x(x-6) = 0$

$x = 0,\ x = 6$

$$Area = \int_0^6 0 - (x^2 - 6x)\, dx$$

$$\int -x^2 + 6x\, dx$$

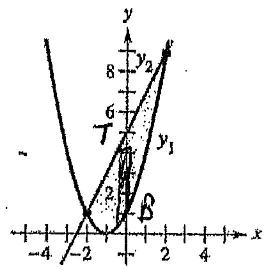
$$-\frac{x^3}{3} + \frac{6x^2}{2}$$

$$\left. -\frac{x^3}{3} + 3x^2 \right]_0^6$$

$$\frac{-6^3}{3} + 3(6)^2 - (0 - 0)$$

$$= \boxed{36}$$

2. $y_1 = x^2 + 2x + 1$
   $y_2 = 2x + 5$




find bounds:

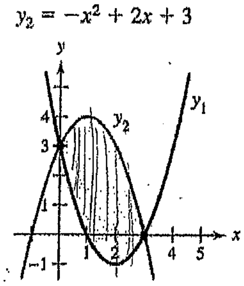
$x^2 + 2x + 1 = 2x + 5$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0 \quad x = 2, -2$

$$Area = \int_{-2}^{2} 2x + 5 - (x^2 + 2x + 1)\, dx$$

3. $y_1 = x^2 - 4x + 3$
   $y_2 = -x^2 + 2x + 3$



$-x^2 + 2x + 3 = x^2 - 4x + 3$

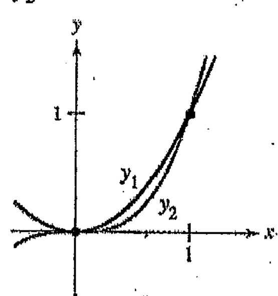
$0 = 2x^2 - 6x$

$0 = 2x(x-3)$

$x = 0, 3$

$$A = \int_0^3 -x^2 + 2x + 3 - (x^2 - 4x + 3)\, dx$$

4. $y_1 = x^2$
   $y_2 = x^3$



bounds: $x^2 = x^3$

$x^2 - x^3 = 0$

$x^2(1 - x) = 0$

$x = 0, 1$

$$A = \int_0^1 x^2 - x^3\, dx$$

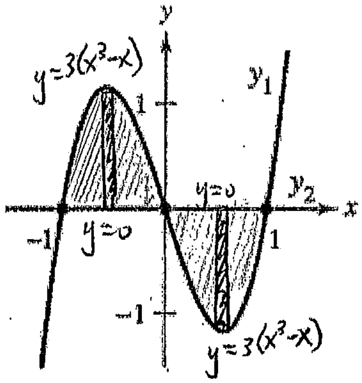
$$= \left. \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1^3}{3} - \frac{1}{4} - (0 - 0)$$

$$= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\,dx$$

**5.** $y_1 = 3(x^3 - x)$

$y_2 = 0$




$$\boxed{A = \int_{-1}^{0} 3(x^3-x) - 0\,dx + \int_{0}^{1} 0 - 3(x^3-x)\,dx}$$

*bounds:

$3(x^3-x) = 0$

$x^3 - x = 0$

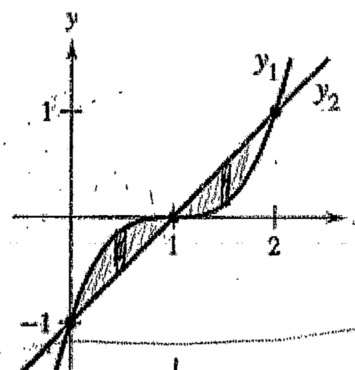
$x(x^2-1) = 0$

$x(x+1)(x-1) = 0$

$x = 0, 1, -1$

**6.** $y_1 = (x-1)^3$

$y_2 = x - 1$




*bounds:

$x - 1 = (x-1)^3$

$x - 1 - (x-1)^3 = 0$

$(x-1)[1-(x-1)^2] = 0$

$(x-1)(1-x^2+2x-1) = 0$

$(x-1)(x)(2-x) = 0$

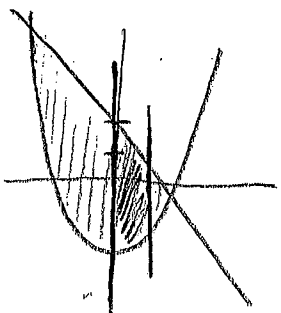
$x = 1, 0, 2$

$$\boxed{Area = \int_{0}^{1} (x-1)^3 - (x-1)\,dx + \int_{1}^{2} x-1-(x-1)^3\,dx}$$

$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\ dx$

**Finding the Area of a Region** In Exercises 17–30, sketch the region bounded by the graphs of the equations and find the area of the region.

**17.** $y = x^2 - 1, \quad y = -x + 2, \quad x = 0, \quad x = 1$



$Area = \int_0^1 -x+2-(x^2-1)\,dx$

$\int -x+2-x^2+1\,dx$

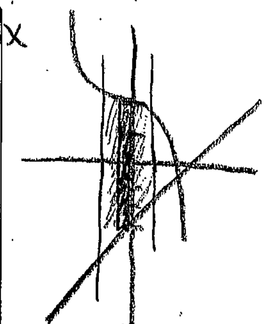
$\int_0^1 -x^2-x+3\,dx$

$= \left.\frac{-x^3}{3} - \frac{x^2}{2} + 3x\right]_0^1$

$= \left(\frac{-1}{3} - \frac{1}{2} + 3\right) - (0 - 0 + 0)$

$= \boxed{\frac{13}{6}}$

**18.** $y = -x^3 + 2, \quad y = x - 3, \quad x = -1, \quad x = 1$



$A = \int_{-1}^{1} -x^3+2-(x-3)\,dx$

$\int -x^3+2-x+3\,dx$

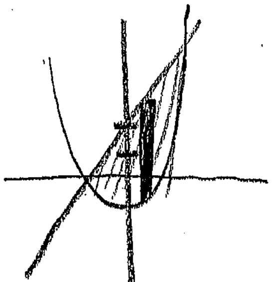
$\int_{-1}^{1} -x^3-x+5\,dx$

$\left.\frac{-x^4}{4} - \frac{x^2}{2} + 5x\right]_{-1}^{1}$

$\left[\frac{-1}{4} - \frac{1}{2} + 5\right] - \left[\frac{-1}{4} - \frac{1}{2} - 5\right]$

$= \boxed{10}$

**19.** $f(x) = x^2 + 2x, \quad g(x) = x + 2$



find intersection:

$x^2 + 2x = x + 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

$A = \int_{-2}^{1} x+2-(x^2+2x)\,dx$

$\int x+2-x^2-2x\,dx$

$\int_{-2}^{1} -x^2-x+2\,dx$

$\left.\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$

$\left(\frac{-1}{3} - \frac{1}{2} + 2\right) - \left(\frac{2^3}{3} - \frac{4}{2} - 4\right)$

$= \boxed{\frac{9}{2}}$

**20.** $y = -x^2 + 3x + 1, \quad y = -x + 1$

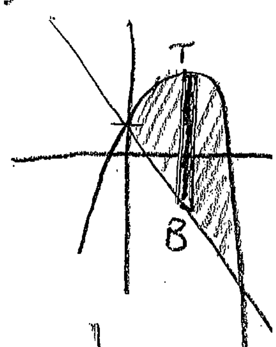
*find intersection:

$-x + 1 = -x^2 + 3x + 1$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$



$A = \int_0^4 -x^2+3x+1-(-x+1)\,dx$

$\int -x^2+3x+1+x-1\,dx$

$\int_0^4 -x^2+4x\,dx$

$\frac{-x^3}{3} + \frac{4x^2}{2}$

$\left.\frac{-x^3}{3} + 2x^2\right]_0^4$

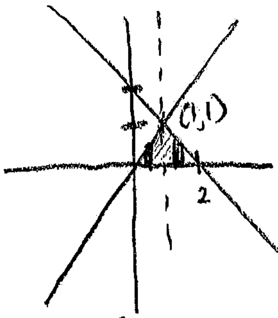
$-\frac{4^3}{3} + 2(4)^2 - \left(\frac{0}{3} + 2(0)^2\right)$

$\frac{-64}{3} + 32 = \boxed{\frac{32}{3}}$

$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$

**Finding the Area of a Region** In Exercises 17–30, sketch the region bounded by the graphs of the equations and find the area of the region.

**21.** $y = x, \quad y = 2 - x, \quad y = 0$



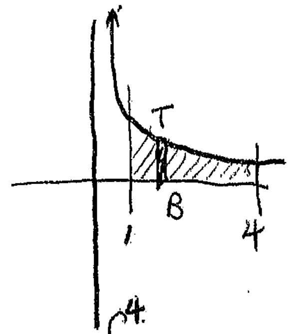

$A = \int_0^1 x - 0\,dx + \int_1^2 2 - x - 0\,dx$

$\left.\frac{x^2}{2}\right]_0^1$ | $\left.2x - \frac{x^2}{2}\right]_1^2$

$\frac{1}{2} - 0$ | $4 - \frac{4}{2} - \left(2 - \frac{1}{2}\right)$

$\frac{1}{2} + \frac{1}{2} = \boxed{1}$

**22.** $y = \frac{4}{x^3}, \quad y = 0, \quad x = 1, \quad x = 4$




$A = \int_1^4 \frac{4}{x^3} - 0\,dx$

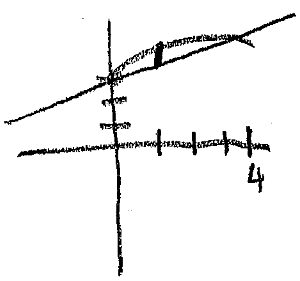
$\int 4x^{-3}\,dx$

$\frac{4x^{-2}}{-2} = \left.\frac{-2}{x^2}\right]_1^4$

$\frac{-2}{4^2} - \frac{-2}{1^2}$

$\frac{-2}{16} + 2 = \boxed{\frac{15}{8}}$

**23.** $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$




*find intersection:

$\sqrt{x} + 3 = \frac{1}{2}x + 3$

$(\sqrt{x})^2 = \left(\frac{x}{2}\right)^2$ | $4x - x^2 = 0$

$x = \frac{x^2}{4}$ | $x(4 - x) = 0$

$4x = x^2$ | $x = 0, 4$

$\int_0^4 \sqrt{x} + 3 - \left(\frac{1}{2}x + 3\right) dx$

$\int \sqrt{x} + 3 - \frac{1}{2}x - 3\,dx$

$\int x^{1/2} - \frac{1}{2}x\,dx$

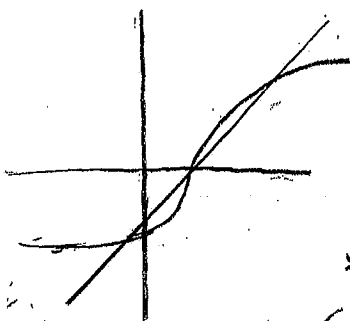
$\frac{x^{3/2}}{3/2} - \frac{1}{2}\left(\frac{x^2}{2}\right)$

$\left.\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4$

$\frac{2}{3}(4)^{3/2} - \frac{4^2}{4} = \frac{2}{3}(8) - 4$

$= \frac{16}{3} - \frac{12}{3} = \boxed{\frac{4}{3}}$

**24.** $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x - 1$



*bounds:

$\sqrt[3]{x-1} = x - 1$

$x - 1 = (x-1)^3$

$x - 1 - (x-1)^3 = 0$

$x - 1\left[1 - (x-1)^2\right] = 0$

$(x-1)(1 - x^2 + 2x - 1) = 0$

$(x-1)(x)(2-x) = 0$

$x = 0, 1, 2$

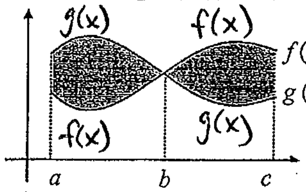
$A = \int_0^1 x - 1 - (x-1)^{1/3} + \int_1^2 (x-1)^{1/3} - (x-1)\,dx = \boxed{\frac{1}{2}}$

**A.P. Calculus AB** **7.1 Worksheet** **No Calculators!**

Key

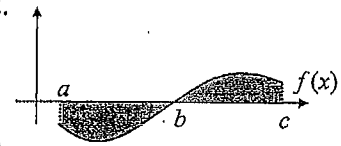
Write an integral that can be used to find the area of the shaded regions.

1.




$$A=\int_a^b g(x)-f(x)\,dx+\int_b^c f(x)-g(x)\,dx$$

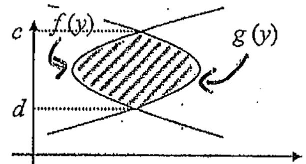
2.




$$A=\int_a^b 0-f(x)\,dx+\int_b^c f(x)-0\,dx$$

$$A=-\int_a^b f(x)\,dx+\int_b^c f(x)\,dx$$

3.

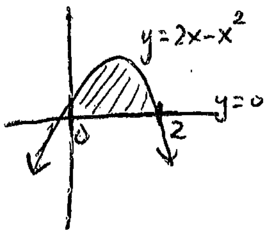



$$A=\int_d^c g(y)-f(y)\,dy$$

Find the area bounded by the regions listed below:

4. the $x$-axis and $y=2x-x^2$

Top-bottom




$$A=\int_0^2 \underbrace{2x-x^2}_{top}-\underbrace{0}_{bottom}\,dx$$

$$\left.\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^2=4-8/3$$

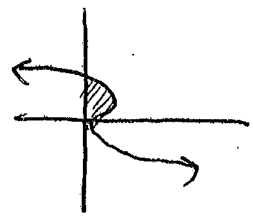
$$=\boxed{4/3 \text{ units}^2}$$

$2x-x^2=0$

$x(2-x)=0$

$x=0,2$

5. the $y$-axis and $x=y^2-y^3$



$y^2(1-y)=0$

$y=0,1$

Right-Left Form

$$A=\int_0^1 \underbrace{y^2-y^3}_{Right}-\underbrace{0}_{Left}\,dy$$

$$\left.\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1$$

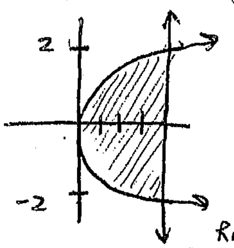
$$\frac{1}{3}-1/4=\boxed{\frac{1}{12}\text{ units}^2}$$

6. $y^2=x$ and $x=4$

Right-Left

set $y^2=4$ $y=\pm 2$

* find intersections



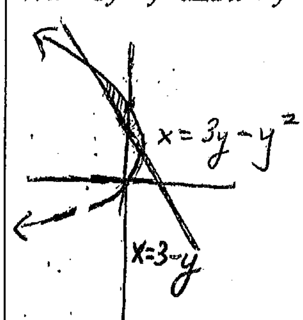

$$A=\int_{-2}^{2} \underbrace{4}_{Right}-\underbrace{y^2}_{Left}\,dy=\left.4y-\frac{y^3}{3}\right]_{-2}^{2}$$

$$=8-\frac{8}{3}-(-8+8/3)$$

$$=16-16/3=\boxed{\frac{32}{3}\text{ units}^2}$$

7. $x=3y-y^2$ and $x+y=3$

$x=3-y$




* find intersections (bounds)

$3-y=3y-y^2$

$y^2-4y+3=0$

$(y-3)(y-1)=0$

$y=1,3$

$$\int_1^3 \underbrace{3y-y^2}_{Right}-\underbrace{(3-y)}_{Left}\,dy$$

$$\int_1^3 3y-y^2-3+y\,dy$$

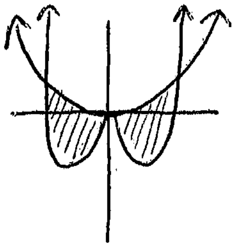
$$\int_1^3 -y^2+4y-3\,dy$$

$$\left.\frac{-y^3}{3}+\frac{4y^2}{2}-3y\right]_1^3$$

$$-\frac{27}{3}+18-9-\left(\frac{-1}{3}+2-3\right)$$

$$=\boxed{4/3\text{ units}^2}$$

8. $y = x^4 - 2x^2$ and $y = 2x^2$



*find intersection:

$x^4-2x^2=2x^2$

$x^4-4x^2=0$

$x^2(x^2-4)=0$

$x=0,2,-2$

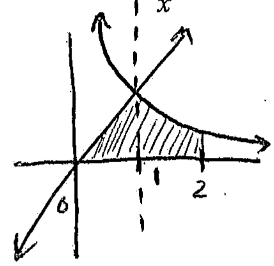
$$A=\int_{-2}^{2} \overbrace{2x^2}^{\text{Top}} - \overbrace{(x^4-2x^2)}^{\text{Bottom}}\,dx$$

$$\int_{-2}^{2} 4x^2-x^4\,dx = \left.\frac{4x^3}{3}-\frac{x^5}{5}\right]_{-2}^{2}$$

$$\frac{32}{3}-\frac{32}{5}-\left(-\frac{32}{3}+\frac{32}{5}\right)$$

$$\frac{64}{3}-\frac{64}{5}=\boxed{\frac{128}{15}\text{ units}^2}$$

9. $y=x$, $y=\frac{1}{x^2}$, $x=2$ $y=0$




*find intersection:

$x=\frac{1}{x^2}$

$x^3=1$, $x=1$

*split into 2 integrals:

$$A=\int_0^1 \overbrace{x}^{\text{top}} - \overbrace{0}^{\text{bottom}}\,dx + \int_1^2 \overbrace{\frac{1}{x^2}}^{\text{top}}-\overbrace{0}^{\text{bottom}}\,dx = \int x^{-2}\,dx$$

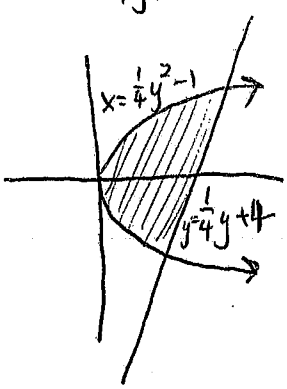
$$\left.\frac{x^2}{2}\right]_0^1 + \left.\frac{x^{-1}}{-1}\right]_1^2 = \left.-\frac{1}{x}\right]_1^2$$

$$\frac{1}{2}-0 + -\frac{1}{2}-\left(-\frac{1}{1}\right)$$

$$\frac{1}{2}-\frac{1}{2}+1=\boxed{1}$$

10. $4x = y^2 - 4$ and $4x = y + 16$ $y=4x-16$

$x=\frac{1}{4}y^2-1$ $x=\frac{1}{4}y+4$




*find intersection:

$$\left[\frac{1}{4}y^2-1=\frac{1}{4}y+4\right]\cdot 4$$

$y^2-4=y+16$

$y^2-y-20=0$

$(y-5)(y+4)=0$

$y=-4,5$

$$A=\int_{-4}^{5} \overbrace{\frac{1}{4}y+4}^{\text{Right}} - \overbrace{\left(\frac{1}{4}y^2-1\right)}^{\text{Left}}\,dy$$

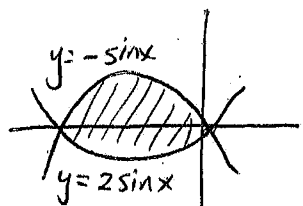
$$\int_{-4}^{5} -\frac{1}{4}y^2+\frac{1}{4}y+5\,dy$$

$$\left.-\frac{y^3}{12}+\frac{y^2}{8}+5y\right]_{-4}^{5}$$

$$\frac{-125}{12}+\frac{25}{8}+25-\left(-\frac{64}{12}+\frac{16}{8}-20\right)$$

$$=\boxed{\frac{243}{3}\text{ units}^2}$$

11. $y=-\sin x$ and $y=2\sin x$, $-\pi \le x \le 0$




*find intersection:

$2\sin x=-\sin x$

$3\sin x=0$

$x=0,-\pi$

$$A=\int_{-\pi}^{0} \overbrace{-\sin x}^{\text{Top}} - \overbrace{2\sin x}^{\text{Bottom}}\,dx = \int_{-\pi}^{0} -3\sin x\,dx$$

$$=\int_{-\pi}^{0} -3\sin x\,dx = \left.3\cos x\right]_{-\pi}^{0} = 3\cos(0)-(3\cos(-\pi))$$

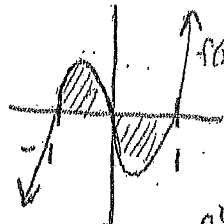
$$=3-(-3)$$

$$=\boxed{6\text{ units}^2}$$

# 7.1 Homework p. 452-453 #1,3,5,17-35 odd, 43, 47, 71

*Area between curves

5) $f(x) = 3(x^3 - x)$    Set up definite integral giving area of region
$g(x) = 0$

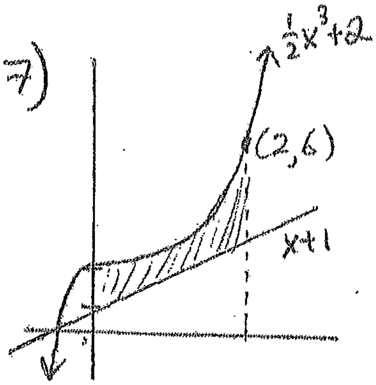



Since $f(x)$ is an odd function, the area regions are equal to each other.

$$A = \int_{-1}^{0} 3(x^3-x)dx + \int_{-1}^{0} 3(x^3-x)dx = 2\int_{-1}^{0} 3(x^3-x)dx$$

$$= \boxed{6\int_{-1}^{0}(x^3-x)\,dx}$$

17)

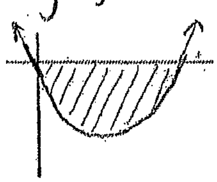



$$A = \int_{0\ \text{left bound}}^{2\ \text{right bound}} \Big[\underbrace{\left(\tfrac{1}{2}x^3+2\right)}_{\text{top function}} - \underbrace{(x+1)}_{\text{bottom function}}\Big]dx = \int \tfrac{1}{2}x^3+2-x-1\,dx$$

$$= \int \tfrac{1}{2}x^3+1-x\,dx = \tfrac{1}{2}\left(\tfrac{x^4}{4}\right)+x-\tfrac{x^2}{2}\Big]_0^2 = \tfrac{2^4}{8}+2-\tfrac{4}{2}-(0+0-0)$$

$$= 2+2-2 = \boxed{2}$$

19) $f(x) = x^2 - 4x$
$g(x) = 0$



* set equations equal to each other to find left and right bounds

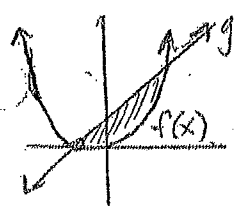
$x^2-4x=0$
$x(x-4)=0$
$x=0,\ x=4$

$$\int_0^4 \underbrace{0}_{\text{top}} - \underbrace{(x^2-4x)}_{\text{bottom}}dx = \int_0^4 -x^2+4x\,dx = -\frac{x^3}{3}+\frac{4x^2}{2}\Big]_0^4$$

$$= -\frac{4^3}{3}+2(4)^2 = \frac{-64}{3}+32$$

$$= \boxed{\frac{32}{3}}$$

21) $f(x) = x^2+2x+1$
$g(x) = 3x+3$




*find left/right bounds:

$x^2+2x+1 = 3x+3$
$x^2-1x-2=0$
$(x-2)(x+1)=0$
$x=2,\ x=-1$

$$\int_{-1}^{2} \underbrace{3x+3}_{\text{top}} - \underbrace{(x^2+2x+1)}_{\text{bottom}}dx$$

$3x+3-x^2-2x-1$
$-x^2+x+2$

$$\int_{-1}^{2} -x^2+x+2\,dx = \frac{-x^3}{3}+\frac{x^2}{2}+2x\Big]_{-1}^{2}$$

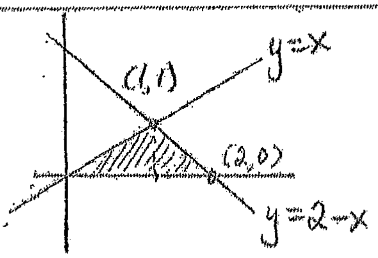
$$\frac{-8}{3}+\frac{4}{2}+4-\left(\frac{1}{3}+\frac{1}{2}-2\right)$$

$$\frac{-8}{3}+2+4-\frac{1}{3}-\frac{1}{2}+2$$

$$= \boxed{\frac{9}{2}}$$

# 7.1 HW (continued)

23) $y=x$, $y=2-x$, $y=0$




$y=x$ and $y=2-x$ intersect at $x=1$

**Method 1**

$$\int_0^1 (\underbrace{x}_{top} - \underbrace{0}_{bottom})dx + \int_1^2 (\underbrace{2-x}_{top} - \underbrace{0}_{bottom})dx$$

$$\left.\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \qquad \left.2x-\frac{x^2}{2}\right]_1^2$$

$$= 4-2-(2-\tfrac{1}{2})$$

$$= \tfrac{1}{2} + \cancel{4} - \cancel{2} - \cancel{2} + \tfrac{1}{2} = \boxed{1}$$

**Method 2**

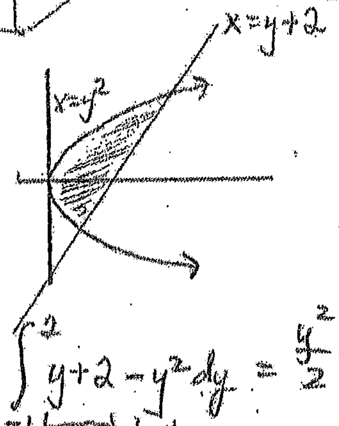
$$\int_{y_1}^{y_2} (\text{Right} - \text{Left})dy \qquad \text{Left: } x=y \qquad \text{Right: } x=2-y$$

$$\int_{0 \text{ lower bound}}^{1 \text{ upper bound}} \underbrace{2-y}_{\text{Right}} - \underbrace{y}_{\text{Left}}\, dy = \int 2-2y = \left. 2y - \frac{2y^2}{2}\right]_0^1$$

$$= 2-1-(0-0) = \boxed{1}$$

27) $f(y)=y^2 \rightarrow x=y^2$

$g(y)=y+2 \quad x=y+2$




$$A = \int_{y_1}^{y_2} \text{Right} - \text{Left}\, dy$$

*find lower/upper bounds

$y^2=y+2 \quad y^2-y+2=0$

$(y-2)(y+1)=0$

$y=2,-1$

$$A = \int_{-1}^{2} \underbrace{y+2}_{\text{Right}} - \underbrace{y^2}_{\text{Left}}\, dy = \left. \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2}$$

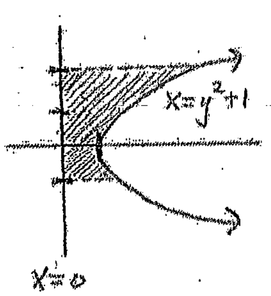
$$= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}$$

$$= 2+4-\frac{9}{3}-\frac{1}{2}+2 = \boxed{\frac{9}{2}}$$

29) $f(y)=y^2+1$

$g(y)=0$, $y=-1$, $y=2 \rightarrow x=y^2+1 \quad y=-1$

$x=0 \quad y=2$




$$\int_{-1}^{2} \overbrace{y^2+1}^{\text{Right}} - \overbrace{0}^{\text{Left}}\, dy$$

$$\left. \frac{y^3}{3} + y \right]_{-1}^{2} = \frac{8}{3} + 2 - \left(\frac{-1}{3} - 1\right)$$

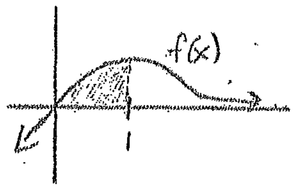
$$= \frac{8}{3} + 2 + \frac{1}{3} + 1 = 3+2+1 = \boxed{6}$$

# 7.1 HW (continued)

47) $f(x) = xe^{-x^2}$

$y = 0$

$0 \le x \le 1$




$A = \int_0^1 \underbrace{xe^{-x^2}}_{\text{top}} - \underbrace{0}_{\text{bottom}}\, dx$

$\int_0^1 xe^{-x^2}\,dx$ $\quad u = -x^2 \quad \frac{du}{dx} = -2x \quad dx = \frac{du}{-2x}$

$\int \cancel{x} e^u \cdot \frac{du}{-2\cancel{x}}$

$\frac{-1}{2}\int e^u\,du$

if $x=0$, $u = -(0)^2 = 0$

if $x=1$, $u = -(1)^2 = -1$

$\frac{-1}{2}e^u \Big]_0^{-1} = \frac{-1}{2}e^{-1} - \left(\frac{-1}{2}e^0\right)$

$= \frac{-1}{2e} + \frac{1}{2} \approx \boxed{0.316}$

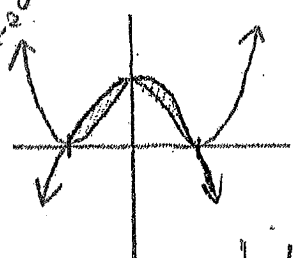
71) The graphs $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ can be found by single integral.

$x^4 - 2x^2 + 1 = 1 - x^2$

$x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$x = 0, 1, -1$



Since $1 - x^2 \geq x^4 - 2x^2 + 1$, $y = 1 - x^2$ will always be the top curve. There is therefore no need to split into 2 integrals.

$A = \int_{-1}^{1} 1 - x^2 - (x^4 - 2x^2 + 1)\,dx$

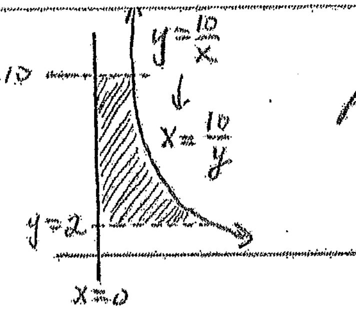
$1 - x^2 - x^4 + 2x^2 - 1 = -x^4 + x^2$

$A = \int_{-1}^{1} x^2 - x^4\,dx = \frac{x^3}{3} - \frac{x^5}{5}\Big]_{-1}^{1} = \frac{1}{3} - \frac{1}{5} - \left(\frac{-1}{3} + \frac{1}{5}\right)$

$= \frac{2}{3} - \frac{2}{5} = \boxed{\frac{4}{15}}$

# 7.1 HW (continued)

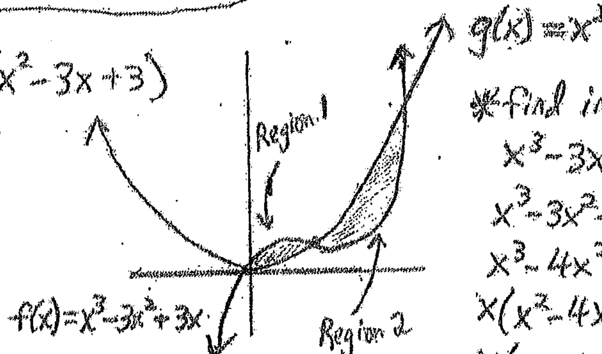
31) $f(x)=\frac{10}{x}$ , $x=0$, $y=2$, $y=10$




$A=\int_{y_1}^{y_2}(\text{Right}-\text{Left})\,dy$

$$\int_2^{10}\underbrace{\frac{10}{y}}_{\text{Right}}-\underbrace{0}_{\text{Left}}\,dy \quad 10\ln|y|\Big]_2^{10} = 10\ln(10)-10\ln(2) = 10\ln\left(\frac{10}{2}\right) = 10\ln 5 \approx \boxed{16.0944}$$

33) $f(x)=x(x^2-3x+3)$
$g(x)=x^2$




*find intersections to determine bounds:

$x^3-3x^2+3x=x^2$

$x^3-3x^2-x^2+3x=0$

$x^3-4x^2+3x=0$

$x(x^2-4x+3)=0$

$x(x-3)(x-1)=0$

$\underline{x=0,\ x=3,\ x=1}$

$$A=\int_0^1 \underbrace{x^3-3x^2+3x}_{\text{top}}-\underbrace{(x^2)}_{\text{bottom}}dx+\int_1^3 \underbrace{x^2}_{\text{top}}-\underbrace{(x^3-3x^2+3x)}_{\text{bottom}}dx = \boxed{\frac{37}{12}}$$

35) $y=x^2-4x+3$
$y=3+4x-x^2$


*find points of intersection:

$x^2-4x+3=-x^2+4x+3$

$2x^2-8x=0$

$2x(x-4)=0$

$\underline{x=0,\ x=4}$

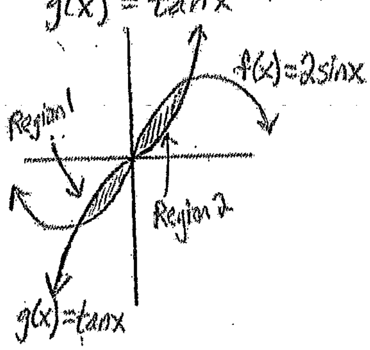
$$A=\int_0^4 -x^2+4x+3-(x^2-4x+3)\,dx$$

$$\int -2x^2+8x\,dx = -\frac{2x^3}{3}+\frac{8x^2}{2}\Big]_0^4$$

$$-\frac{2}{3}x^3+4x^2\Big]_0^4 = -\frac{128}{3}+64-(0+0)$$

$$=\boxed{\frac{64}{3}}$$

43) $f(x)=2\sin x$ $\quad -\frac{\pi}{3}\le x\le\frac{\pi}{3}$
$g(x)=\tan x$




*Region 1 = Region 2

$$A=2\cdot\int_0^{\pi/3} 2\sin x-\tan x\,dx$$

$$=2\cdot(-2\cos x+\ln|\cos x|)\Big]_0^{\pi/3}$$

$=2(-2\cos(\pi/3)+\ln|\cos(\pi/3)|)-2(-2(1)+0)$

$=-4\left(\frac{1}{2}\right)+2\ln\left|\frac{1}{2}\right|+4$

$=-2+4+2\ln(0.5)$

$=2+2\ln(0.5)\approx\boxed{0.614}$

Key

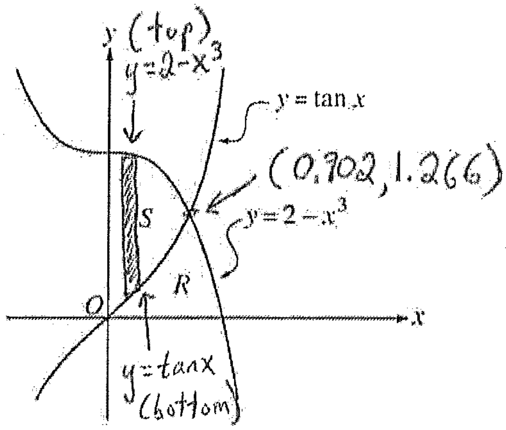
1. Let $R$ and $S$ be the regions in the first quadrant shown in the figure above. The region $R$ is bounded by the $x$-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region $S$ is bounded by the $y$-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

(a) Find the area of $S$

$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)dx \qquad \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

(in the forms of " y = __ ") (in the form of " x = __ ")

i) (Top – Bottom Method)

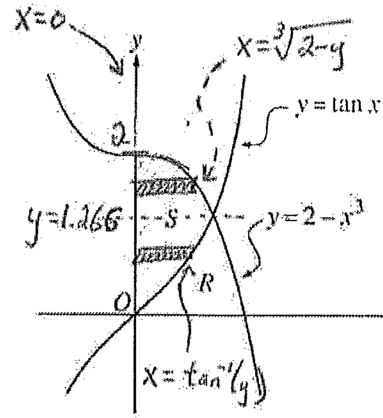



$$Area = \int_0^{0.902} 2 - x^3 - (\tan x)\,dx$$

Top – Bottom

$$\boxed{Area = 1.161}$$

ii) (Right – Left Method)




$y = \tan x \rightarrow x = \arctan(y)$ (or $x = \tan^{-1}(y)$)

$x^3 = 2 - y$

$x = \sqrt[3]{2-y}$ or ($x = (2-y)^{1/3}$)

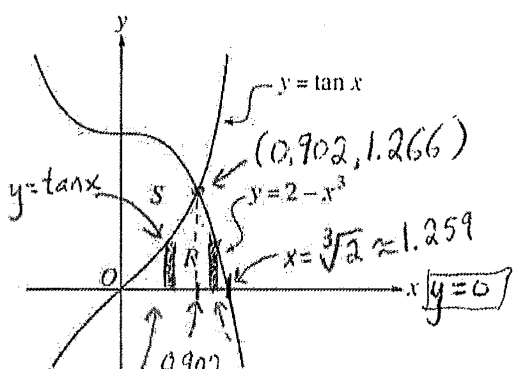
$$\int_0^{1.266} \tan^{-1}(y) - 0\,dy + \int_{1.266}^{2} \sqrt[3]{2-y} - 0\,dy$$

(Right) – (Left) (Right) – (Left)

$$Area = 0.664 + 0.4965 = \boxed{1.161}$$

(b) Find the area of $R$

i) (Top – Bottom Method)

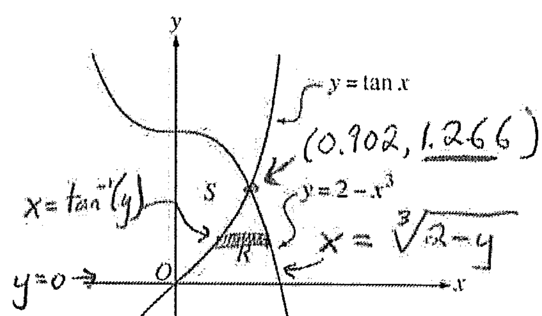



$$Area = \int_0^{0.902} \tan x - 0\,dx + \int_{0.902}^{1.259} 2 - x^3 - 0\,dx$$

(top) – (bottom) (top) – (bottom)

$$Area = 0.478 + 0.251 = \boxed{0.729}$$

ii) (Right – Left Method)




$$Area = \int_0^{1.266} \sqrt[3]{2-y} - \tan^{-1}(y)\,dy$$

(Right) – (Left)

$$\boxed{Area = 0.729}$$

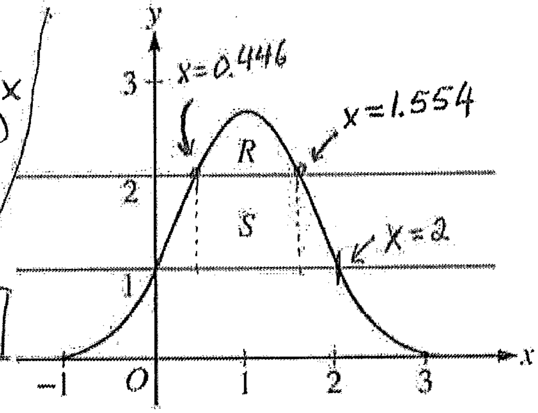
**2)** Let $R$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let $S$ be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of $R$.

(b) Find the area of $S$.

a) $\text{Area}_{(R)} = \int_{0.446}^{1.554} e^{2x-x^2} - 2\, dx$ (top) − (bottom)

$\boxed{\text{Area}_{(R)} = 0.514}$




b) option 1:

$\int_{0}^{0.446} e^{2x-x^2} - 1\, dx + \int_{0.446}^{1.554} 2 - 1\, dx + \int_{1.554}^{2} e^{2x-x^2} - 1\, dx = \boxed{1.546}$

option 2: $\int(R+S) - \int R$

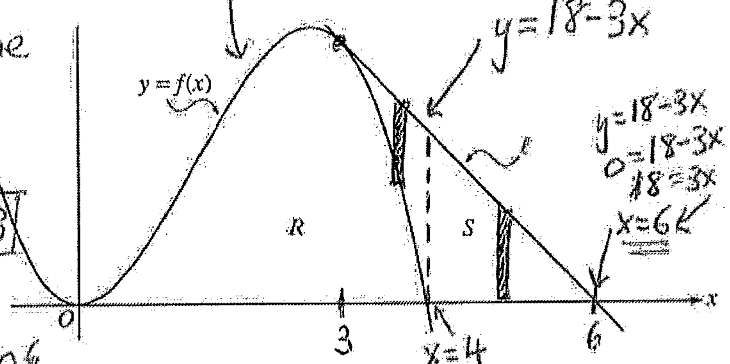
$\int_{0}^{2} e^{2x-x^2} - 1\, dx - \int_{0.446}^{1.554} e^{2x-x^2} - 2\, dx = \boxed{1.546}$

(top) − (bottom)

**3)** Let $f$ be the function given by $f(x) = 4x^2 - x^3$, and let $\ell$ be the line $y = 18 - 3x$, where $\ell$ is tangent to the graph of $f$. Let $R$ be the region bounded by the graph of $f$ and the $x$-axis, and let $S$ be the region bounded by the graph of $f$, the line $\ell$, and the $x$-axis, as shown above.

(a) Show that $\ell$ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of $S$.




a) * Show that the graph f(x) has same slope as line $y = 18 - 3x$ at $x = 3$

* slope of graph: $f'(x) = 8x - 3x^2$

$f'(3) = 8(3) - 3(3)^2 = \boxed{-3}$

slope of line:

$y = -3x + 18 \rightarrow \boxed{m = -3}$ ← same slope

$y = 18 - 3x$
$0 = 18 - 3x$
$18 = 3x$
$x = 6$

b) $\text{Area}_{(S)} = \int_{3}^{4} 18 - 3x - (4x^2 - x^3)\, dx + \int_{4}^{6} 18 - 3x - 0\, dx$

(Top) − (Bottom) (Top) − (Bottom)

$0 = 4x^2 - x^3$
$0 = x^2(4 - x)$
$x = 4$

$\text{Area}(S) = 1.917 + 6 = \boxed{7.917}$

**4)** Let $f$ be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let $R$ be the shaded region in the second quadrant bounded by the graph of $f$, and let $S$ be the shaded region bounded by the graph of $f$ and line $\ell$, the line tangent to the graph of $f$ at $x = 0$, as shown above.

$f(0) = 3$ $f'(0) = \frac{-1}{2}$

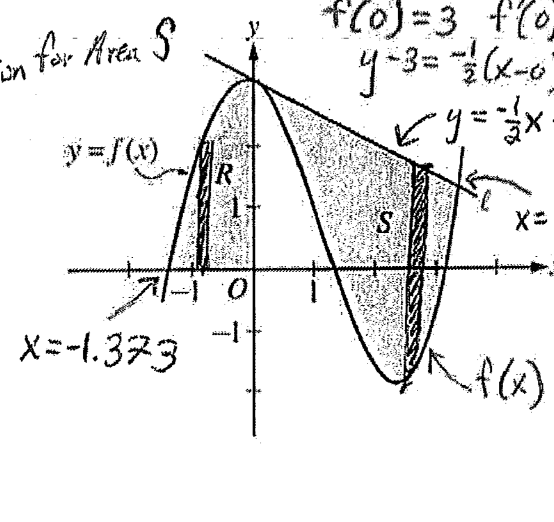
$y - 3 = \frac{-1}{2}(x - 0)$

$y = \frac{-1}{2}x + 3$

(a) Find the area of $R$.

b) Write integral expression for Area S

a) $\int_{-1.373}^{0} f(x) - 0\, dx = \boxed{2.903}$

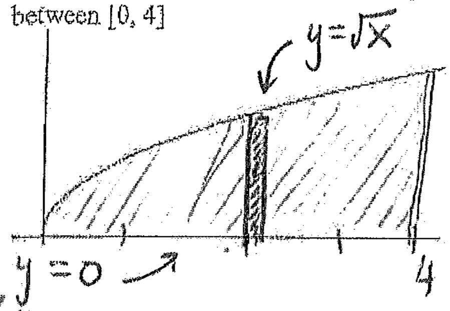



b) Area of S: $\boxed{\int_{0}^{3.389} \frac{-1}{2}x + 3 - f(x)\, dx}$

(top) − (bottom)

$\frac{x^{3/2}}{3/2}$ $3^{rd}$

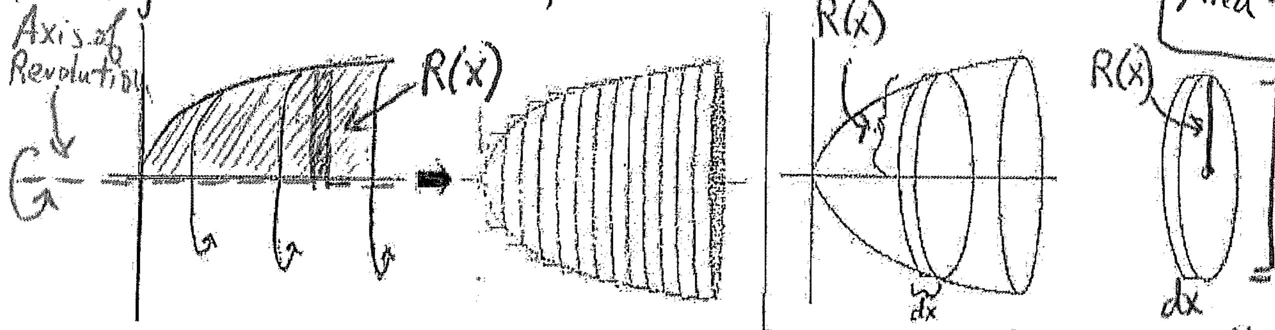
Recall, finding area under the curve $y = \sqrt{x}$ between [0, 4]




$$Area = \int_0^4 \underbrace{\sqrt{x} - 0}_{\text{height}} \underbrace{dx}_{\text{width}} \rightarrow \int x^{1/2} dx \quad \frac{2}{3}x^{3/2}\Big]_0^4$$

$$\frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}$$

$$\boxed{Area = \frac{16}{3} \text{ units}^2}$$




$Area = \pi r^2$

$\pi[R(x)]^2$

Radius $[R(x)]$ = *distance from the AOR (Axis of Revolution) to the graph curve*

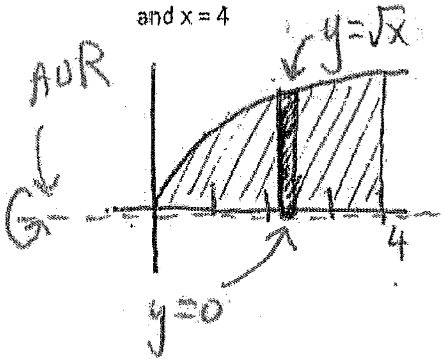
$$Volume = \int_{x_1}^{x_2} \underbrace{\pi[R(x)]^2}_{\text{Area}} \underbrace{dx}_{\text{width}}$$

***Disc Method***: $Volume = \pi \int_{x_1}^{x_2} [R(x)]^2 dx$

(Top-Bottom) "y = _"

AOR: $y=0$

Example 1: Find the volume of the solid formed by rotating the graph of $y = \sqrt{x}$ around the x-axis between x = 0 and x = 4




$R(x) = \sqrt{x} - 0$

$R(x) = \sqrt{x}$

$$V = \pi \int_0^4 [\sqrt{x}]^2 dx \rightarrow \pi \int_0^4 x\,dx \rightarrow \frac{x^2}{2}\Big]_0^4$$

$$\frac{4^2}{2} - \frac{0^2}{2} = \boxed{8\pi \text{ units}^3}$$

$Radius\ [R(x)] = distance\ from\ the\ AOR\ (Axis\ of\ Revolution)\ to\ the\ graph\ curve$

$$\textbf{Disc Method}:\ Volume = \pi \int_{x_1}^{x_2} [R(x)]^2\,dx$$

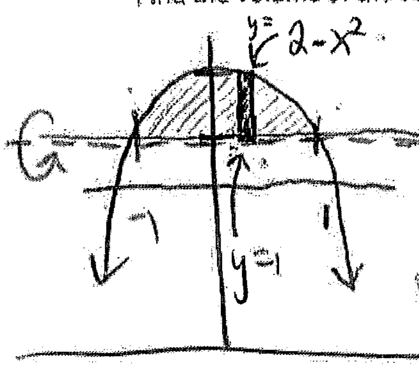
---

Example 2:

Find the volume of the solid created by $f(x) = 2 - x^2$ revolved around the line $y = 1$ and $y=1$ AOR

$y = 2 - x^2$

$y = -x^2 + 2$




$R(x) = 2 - x^2 - (1) = 1 - x^2$

* find bounds (intersections)

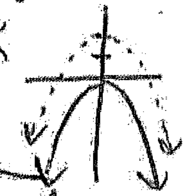
$1 = 2 - x^2$

$\sqrt{x^2} = \pm\sqrt{1}$

$x = -1$

$x = 1$

$$V = \pi \int_{-1}^{1} \left[1 - x^2\right]^2 dx$$



$$\boxed{V = \frac{16}{15}\pi \text{ units}^3}$$

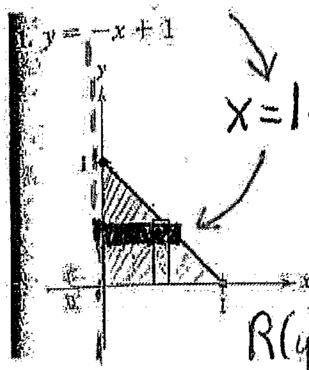
7.2a Disc Method Classwork Problems

**Finding the Volume of a Solid** In Exercises 1–6, set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the ~~x-axis~~ y-axis AOR

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2 dy$$

(Right-Left) "x = __"

1. $y = -x + 1$



$x = 1 - y$

Right − Left

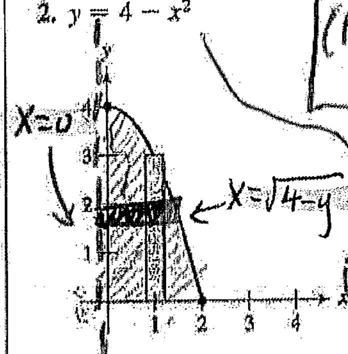
$R(y) = 1 - y - 0$

$R(y) = 1 - y$

$$V = \pi \int_0^1 [1 - y]^2 dy$$

$$\boxed{V = \frac{1}{3}\pi \text{ units}^3}$$

2. $y = 4 - x^2$




$y = 4 - x^2$

$\sqrt{x^2} = \pm\sqrt{4 - y}$

$R(y) = \sqrt{4 - y} - 0$

$R(y) = \sqrt{4 - y}$

$$V = \pi \int_0^4 \left[\sqrt{4 - y}\right]^2 dy$$

$$\boxed{V = 8\pi \text{ units}^3}$$

# 7.2a Disc Method Practice Problems Worksheet

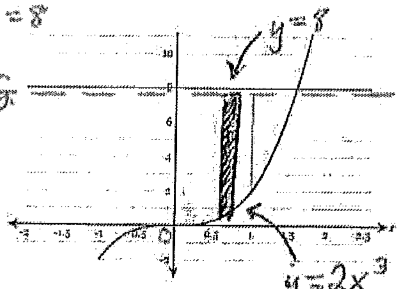
Key

| Disc Method: (Top – Bottom) | Disc Method: (Right – Left) |
|---|---|
| $V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$ | $V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$ |
| (expression(s) used above has form: " y = ____ ") | (expression(s) used above has form: " x = ____ ") |

1. Let the region R be the area enclosed the function $f(x) = 2x^3$ the horizontal line y=8, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 8

AOR $y=8$




$R(x) = 8 - 2x^3$

* intersection:

$2x^3 = 8$

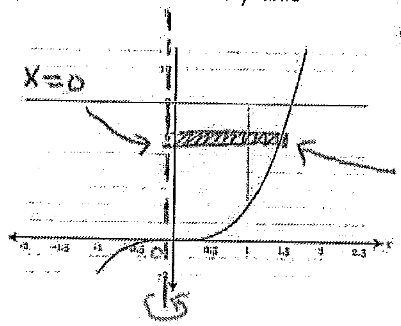
$x^3 = 4$

$x = \sqrt[3]{4} \approx 1.587$

$V = \pi \int_0^{1.587} [8 - 2x^3]^2\, dx$

$\boxed{V = 65.310\pi \text{ units}^3}$

b) rotated about the y-axis




$y = 2x^3$

$\frac{y}{2} = x^3$

$\sqrt[3]{\frac{y}{2}} = x$

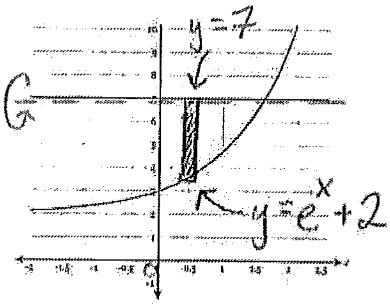
$R(y) = \sqrt[3]{\frac{y}{2}} - 0 = \sqrt[3]{\frac{y}{2}}$

$V = \pi \int_0^8 \left[\sqrt[3]{\frac{y}{2}}\right]^2 dy = \boxed{12.095\pi \text{ units}^3}$

2) Let the region R be the area enclosed the function $f(x) = e^x + 2$, the horizontal line y=7, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 7

$y=7$




$V = \pi \int_0^{\ln 5} [5 - e^x]^2\, dx$

$\boxed{V = 12.236\pi \text{ units}^3}$

$R(x) = 7 - (e^x + 2) = 5 - e^x$

* intersection:

$e^x + 2 = 7$

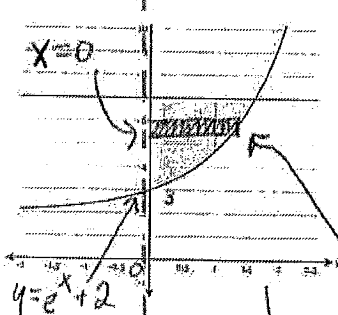
$e^x = 5$

$\ln e^x = \ln 5$

$x \cancel{\ln e} = \ln 5$

$x = \ln 5$

b) rotated about the y-axis




$y = e^x + 2$

$y = e^0 + 2$

$y = 1 + 2 = 3$

x=0

$y = e^x + 2$

$y - 2 = e^x$

$\ln(y-2) = \ln e^x$

$\ln(y-2) = x \cdot \cancel{\ln e}$

$x = \ln(y-2)$

$V = \pi \int_3^7 [\ln(y-2)]^2\, dy$

$R(y) = \ln(y-2) - 0$

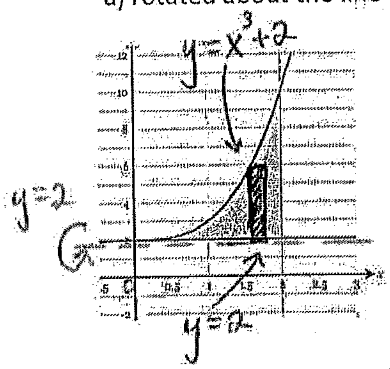
$R(y) = \ln(y-2)$

$\boxed{V = 4.857\pi \text{ units}^3}$

| Disc Method: (Top – Bottom) | Disc Method: (Right – Left) |
|---|---|
| $V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$ | $V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$ |
| (expression(s) used above has form: " y = ____ ") | (expression(s) used above has form: " x = ____ ") |

3) Let the region R be the area enclosed by the function $f(x) = x^3 + 2$, the horizontal line y=2, and the vertical lines x=0 and x=2. Find the volume of the solid generated when shaded region is:

a) rotated about the line y = 2




$R(x) = x^3+2-(2) = x^3$

$V = \pi \int_0^2 [x^3]^2\, dx$

$$\boxed{V = \frac{128}{7}\pi \text{ units}^3}$$

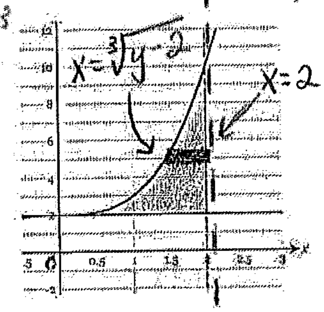
b) rotated about x = 2

*intersection:

$(\sqrt[3]{y-2})^3 = (2)^3$

$y-2=8$

$y=10$




$f(x) = x^3+2$

$y = x^3+2$

$y-2 = x^3$

$\sqrt[3]{y-2} = \sqrt[3]{x^3}$

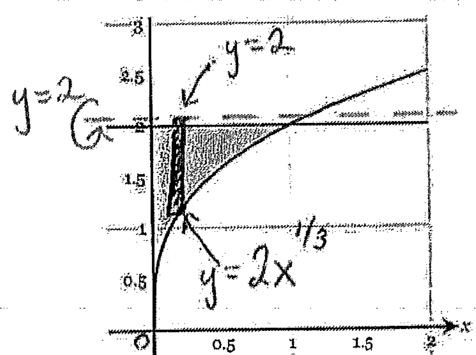
$\sqrt[3]{y-2} = x$

$R(y) = 2-\sqrt[3]{y-2}$

$$V = \pi \int_2^{10} \left[2-\sqrt[3]{y-2}\right]^2 dy = \boxed{3.199\pi \text{ units}^3}$$

4. Let the region R be the area enclosed the function $f(x) = 2x^{\frac{1}{3}}$, the horizontal line y=2, and the y-axis. Find the volume of the solid generated when shaded region is

a) rotated about the line y = 2




$R(x) = 2-2x^{1/3}$

$V = \pi \int_0^1 \left[2-2x^{1/3}\right]^2 dx$

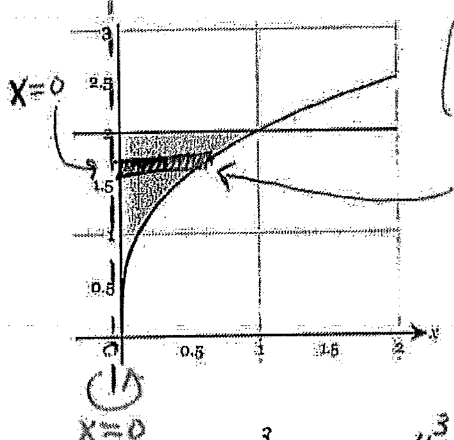
$$\boxed{V = 0.399\pi \text{ units}^3}$$

b) rotated about y-axis

$y = 2x^{1/3}$

$\left(\frac{y}{2}\right)^3 = \left(x^{1/3}\right)^3$

$\frac{y^3}{8} = x$




$R(y) = \frac{y^3}{8} - 0 = \frac{y^3}{8}$

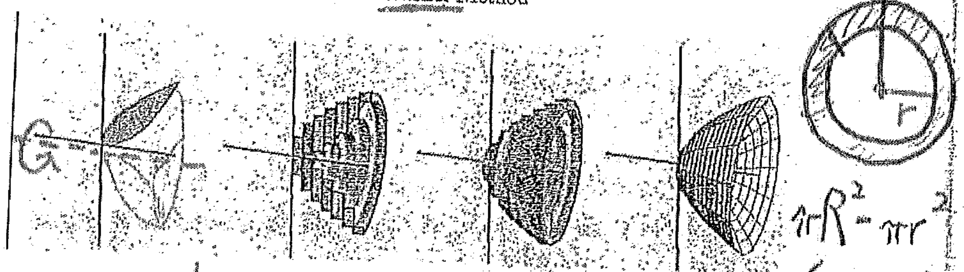
$$V = \pi \int_0^2 \left(\frac{y^3}{8}\right)^2 dy = \boxed{\frac{2}{7}\pi \text{ units}^3}$$

Non-AP Calculus Ch. 7.2b Volume: Washer Method Notes and Classwork problems circular rings

Reviewing Disc Method

$$V = \pi\int_{x_1}^{x_2} [R(x)]^2\,dx$$

Illustration of Washer Method



R r

$\pi R^2 - \pi r^2$

$\pi(R^2 - r^2)$

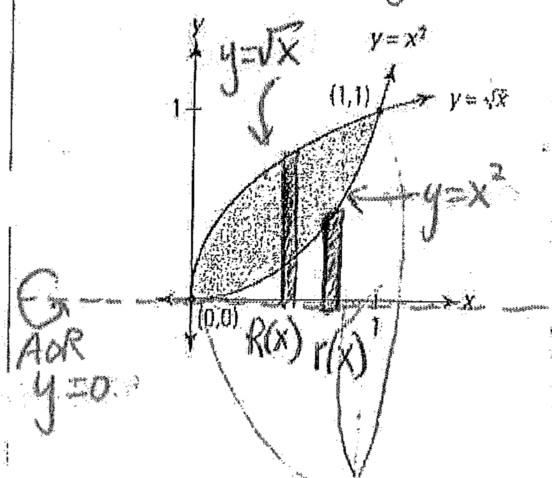
*Radius* $[R(x)]$ = *distance from the AOR (Axis of Revolution) to the further graph curve* center

*radius* $[r(x)]$ = *distance from the AOR (Axis of Revolution) to the closer graph curve*

*Washer Method: Volume* $= \pi\int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\,dx$ (Top - Bottom)

$$V = \pi\int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\,dx$$

**Example 1:** Find the volume of the solid created enclosed region of $y = x^2$ and $y = \sqrt{x}$ revolving about the x-axis (AOR $y=0$)



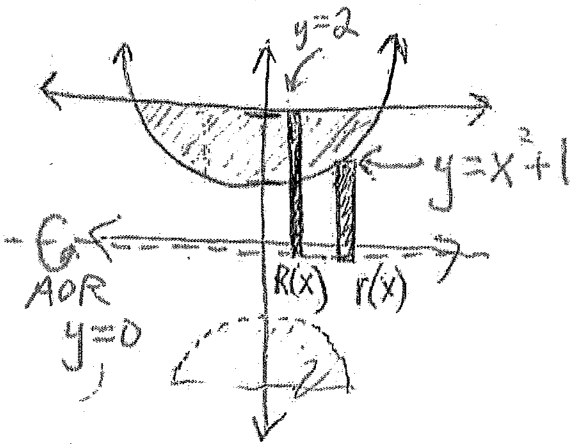

$R(x) = \sqrt{x} - (0) \rightarrow \sqrt{x}$

$r(x) = x^2 - (0) \rightarrow x^2$

$$V = \pi\int_0^1 [\sqrt{x}]^2 - [x^2]^2\,dx$$

$$\boxed{V = 0.3\pi \text{ or } \frac{3}{10}\pi \text{ units}^3}$$

**Example 2**: Find the volume of the solid created by revolving the function $y = x^2 + 1$ bounded by the line $y = 2$ revolved about the x-axis. → AOR $y=0$




$R(x) = 2 - (0) \rightarrow 2$

$r(x) = x^2+1 - (0) \rightarrow x^2+1$

* find intersection (bounds)

$x^2 + 1 = 2$

$\sqrt{x^2} = \pm\sqrt{1}$

$x = 1, x = -1$

$$V = \pi\int_{-1}^{1} [2]^2 - [x^2+1]^2\,dx$$

$$\boxed{V = \frac{64}{15}\pi \text{ units}^3}$$

$Radius\ [R(x)] = distance\ from\ the\ AOR\ (Axis\ of\ Revolution)\ to\ the\ \mathbf{further}\ graph\ curve$

$radius\ [r(x)] = distance\ from\ the\ AOR\ (Axis\ of\ Revolution)\ to\ the\ \mathbf{closer}\ graph\ curve$

$$Washer\ Method\!: Volume = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 dx$$

---

**Example 3**: Find the volume of the solid created by revolving the function $y = x^2 + 1$ bounded by the line $y = 2$ and the y-axis about the line $y = 4$

---

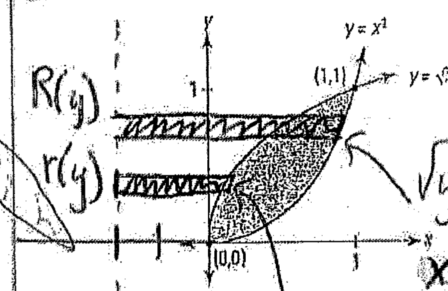
Washer Method (Right − Left)

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$$

needs the form "$x =$ ___"

Change Problem

**Example 4**: Find the volume of the solid created enclosed region of $y = x^2$ and $y = \sqrt{x}$ revolving about the line ~~y = 1~~ Revolve about $x = -2$




$\sqrt{y} = \sqrt{x^2}$

$x = \sqrt{y}$

$(y)^2 = (\sqrt{x})^2$

$x = y^2$

AOR

$x = -2$

$R(y) = \sqrt{y} - (-2) \rightarrow \sqrt{y} + 2$

$r(y) = y^2 - (-2) \rightarrow y^2 + 2$

$$V = \pi \int_0^1 [\sqrt{y} + 2]^2 - [y^2 + 2]^2 dy$$

$$\boxed{V = 1.633\pi \text{ units}^3}$$

or $5.131$ units$^3$

# 7.2b Volume - Washer Method Practice Problems Worksheet

Key

| Washer Method: (Top – Bottom) – Vertical Radius | Washer Method: (Right – Left) – Horizontal Radius |
|---|---|
| $V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$ | $V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$ |
| (expression(s) used above has form: " y = ___ ") | (expression(s) used above has form: " x = ___ ") |

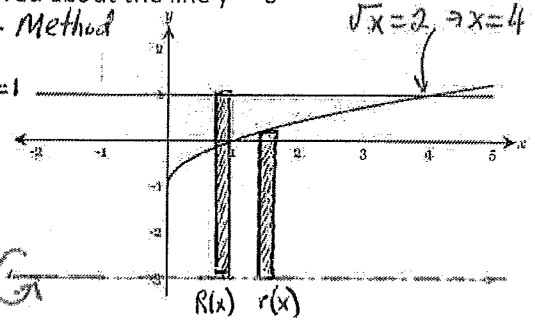
1) Let the region R be the area enclosed the the function $f(x) = \sqrt{x} - 1$, the horizontal line y=1, and the $y$-axis. Find the volume of the solid generated when the region is:

a) revolved about the line y = -3

$\sqrt{x} - 1 = 1$

$\sqrt{x} = 2 \Rightarrow x = 4$

*Washer Method




$R(x) = 1 - (-3) = 4$

$r(x) = \sqrt{x} - 1 - (-3) = \sqrt{x} + 2$

$$V = \pi \int_0^4 [4]^2 - [\sqrt{x}+2]^2\, dx$$

$\boxed{V = 18.667\pi \text{ units}^3}$

b) revolved about the line x = - 1

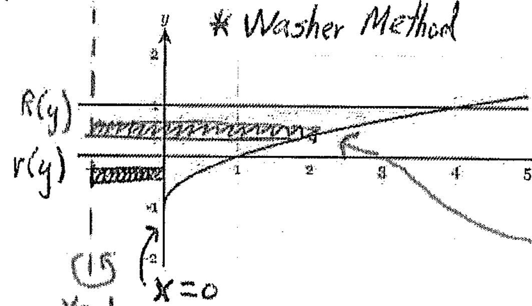
*Washer Method

$y = \sqrt{x} - 1$

$y + 1 = \sqrt{x}$

$(y+1)^2 = (\sqrt{x})^2$

$x = (y+1)^2$




$R(y) = (y+1)^2 - (-1) = (y+1)^2 + 1$

$r(y) = 0 - (-1) = 1$

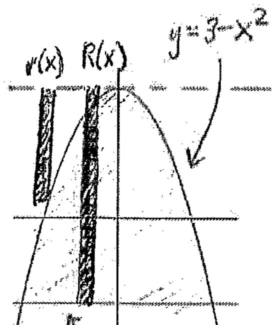
$$V = \pi \int_{-1}^{1} [(y+1)^2 + 1]^2 - [1]^2\, dy$$

$\boxed{V = \frac{176}{15}\pi \text{ units}^3}$

2) Let the region R be the area enclosed the the function $f(x) = 3 - x^2$ the line y = -2. Find the volume of the solid generated when the region is:

a) revolved about the line y = 3

*Washer Method




$R(x) = 3 - (-2) = 5$

$r(x) = 3 - (3 - x^2) = x^2$

*Find intersection (bounds)

$3 - x^2 = -2$

$5 = x^2$

$\sqrt{5} = \sqrt{x^2}$

$\pm\sqrt{5} = x$

$$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} (5)^2 - (x^2)^2\, dx$$

$\boxed{V = 89.443\pi \text{ units}^3}$

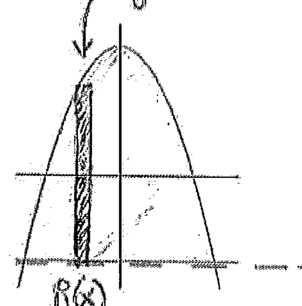
b) revolved about the line y = - 2

*Disc Method

$R(x) = 3 - x^2 - (-2)$

$R(x) = 5 - x^2$

$$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} [5 - x^2]^2\, dx$$




$\boxed{V = 59.628\pi \text{ units}^3}$

*intersections:

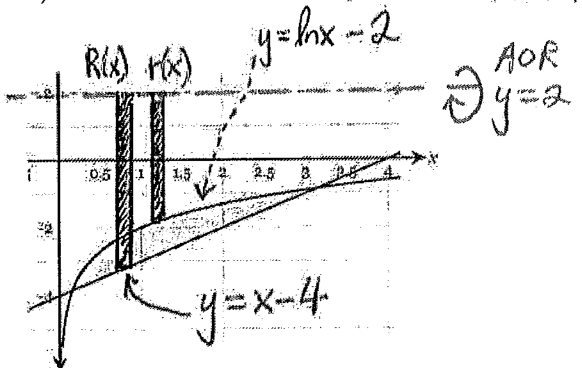
$3 - x^2 = -2$

$x = \pm\sqrt{5}$

| **Washer Method: (Top – Bottom) – Vertical Radius** | **Washer Method: (Right – Left) – Horizontal Radius** |
| --- | --- |
| $V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$ | $V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$ |
| (expression(s) used above has form: " y = ____ ") | (expression(s) used above has form: " x = ____ ") |

3) Let the region R be the area enclosed the function $f(x) = \ln x - 2$ and $g(x) = x - 4$. Find the volume of the solid generated when the region is:

a) revolved about the line $y = 2$ *washer Method




$R(x) = 2-(x-4) = 2-x+4 = 6-x$

$r(x) = 2-(\ln x-2) = 2-\ln x+2 = 4-\ln x$

*find bounds:

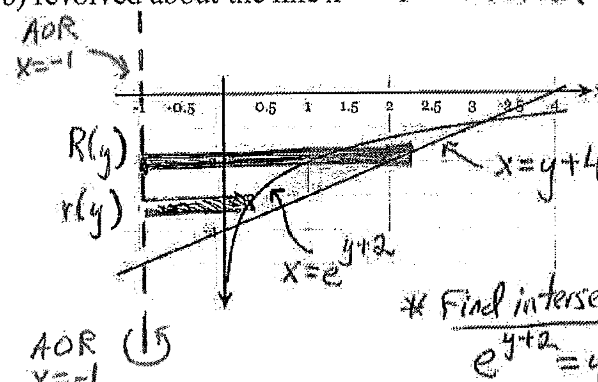
set $\ln x - 2 = x - 4$

$x = 0.1586,\ x = 3.146$

$$V = \pi \int_{0.159}^{3.146} (6-x)^2 - (4-\ln x)^2 \, dx$$

$$\boxed{V = 16.402\pi \text{ units}^3}$$

b) revolved about the line x = –1 *washer Method




*Rewrite equations:

$y = \ln x - 2$ | $y = x-4$

$y+2 = \ln x$ | $y+4 = x$

$e^{y+2} = e^{\ln x}$

$e^{y+2} = x$

$x = e^{y+2}$

*Find intersections:

$e^{y+2} = y+4$

$y = -0.853,\ y = -3.841$

$R(y) = y+4-(-1)$
$= y+5$

$r(y) = e^{y+2} - (-1)$
$= e^{y+2}+1$

$$V = \pi \int_{-3.841}^{-0.853} (y+5)^2 - (e^{y+2}+1)^2 \, dy$$

$$\boxed{V = 9.341\pi \text{ units}^3}$$

4) Let the region R be the area enclosed by the function $f(x) = x^2 + 2$, the horizontal line y=2, & the vertical lines x=0 & x=4. Find volume of the solid generated when region is:

a) revolved about the line $x = 5$

*washer Method

*rewrite equation:

$y = x^2+2$

$y-2 = x^2$

$\pm\sqrt{y-2} = \sqrt{x^2}$

$x = \sqrt{y-2}$

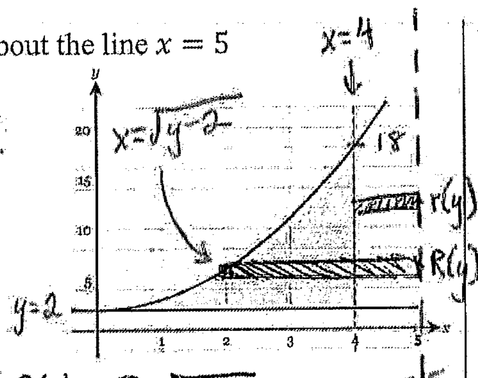
*find intersection:

$\sqrt{y-2} = 4$

$(\sqrt{y-2})^2 = (4)^2$

$y-2 = 16$

$y = 18$




$R(y) = 5-\sqrt{y-2}$

$r(y) = 5-(4) = 1$

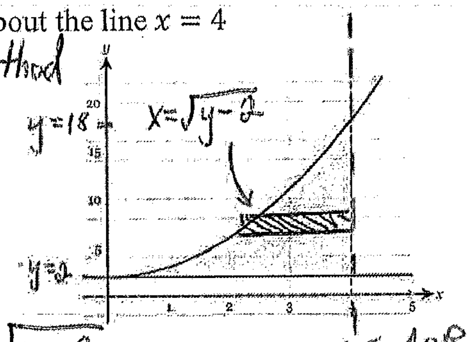
AOR x=5

$$V = \pi \int_2^{18} [5-\sqrt{y-2}]^2 - [1]^2 dy$$

$$\boxed{V = 85.333\pi \text{ units}^3}$$

b) revolved about the line $x = 4$

*Disc Method




$R(y) = 4-\sqrt{y-2}$

AOR x=4

$$V = \pi \int_2^{18} [4-\sqrt{y-2}]^2 dy$$

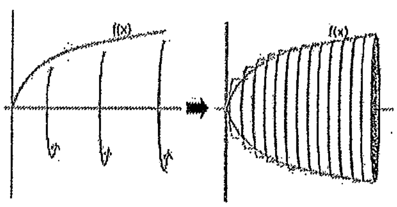
$$\boxed{V = 42.667\pi \text{ units}^3}$$

# AP Calculus Ch. 7.2c Volumes with Known Cross Section

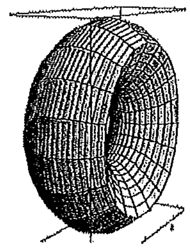
Key

Cross-section is the shape that results if we cut the object in half and look at the resulting shape sideways:

Disc Method



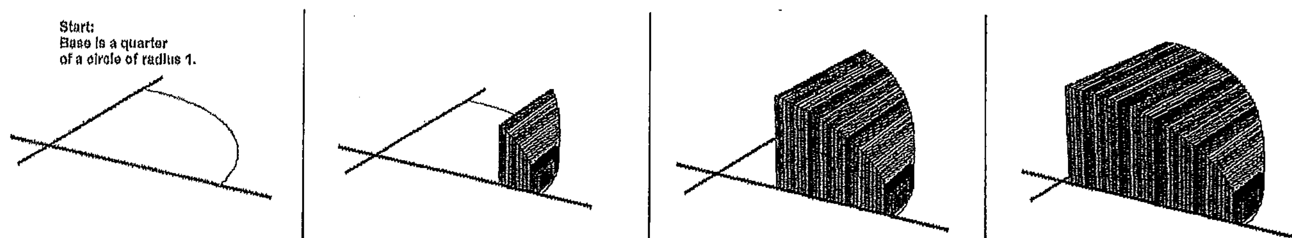

Washer Method



$V = \pi \int [Area \text{ of cross section}]\, dx$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR), resulting in circular Area, either $Area = \pi[R(x)]^2$ or $Area = \pi[R(x)]^2 - \pi[r(x)]^2$

Now, we still have a shaded region, but now it will act as the base of the 3D object. We will now build the cross-section area on top of this base. (no longer rotating around an axis)

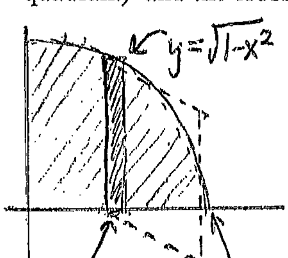



| Top-Bottom Vertical base | Right-Left Horizontal base |
|---|---|
| $V = \int_{x_1}^{x_2} [Area \text{ of cross section}]\, dx$ | $V = \int_{y_1}^{y_2} [Area \text{ of cross section}]\, dy$ |
| *Note: All values in integral are in terms of x (in the form of " y = ____ " ) | *Note: All values in integral are in terms of y (in the forms of " x = ______ " ) |

**Areas formulas for for Cross- sections:**

| | | |
|---|---|---|
| 1. Square: $A = (base)^2$ | 2. Isosceles Right Triangle (leg on base): $A = \frac{1}{2}(base)^2$ | 3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$ |
| 4. Rectangle: A = (base)(height) | 5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$ | 6. Semicircle: $A = \frac{\pi}{8}(base)^2$ |

**Example 1:** Find the volume of the solid if the base is bounded by curve y = $\sqrt{1-x^2}$ , y = 0, x = 0 ( in the first quadrant) and the cross sections are squares parallel to the y-axis.




base $= \sqrt{1-x^2} - 0 = \sqrt{1-x^2}$
(Top-Bottom)

$Area = (base)^2 = (\sqrt{1-x^2})^2$

intersection:
$\sqrt{1-x^2} = 0$ | $x^2 = 1$
$1 - x^2 = 1$ | $x = 1$

$V = \int_{x_1}^{x_2} (Area)\, dx$

$V = \int_0^1 [\sqrt{1-x^2}]^2 dx$

$\boxed{V = \frac{2}{3} \text{ units}^3}$

| Top-Bottom Vertical base | Right-Left Horizontal base |
|---|---|
| $V = \int_{x_1}^{x_2} [Area \text{ of cross section}]dx$ | $V = \int_{y_1}^{y_2} [Area \text{ of cross section}]dy$ |
| *Note: All values in integral are in terms of x (equations in the form of " y = ____ ") | *Note: All values in integral are in terms of y (equations in the form of " x = ____ ") |

**Areas formulas for for Cross- sections:**

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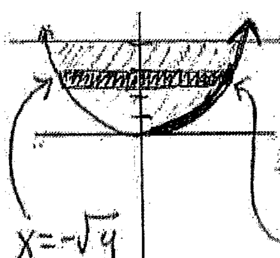
| 4. Rectangle: A = (base)(height) | 5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$ | 6. Semicircle: $A = \frac{\pi}{8}(base)^2$ |

**Example 2:** Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are isosceles right triangles whose hypotenuse lie on the base and are parallel to the x-axis.




base (right-left): $\sqrt{y} - (-\sqrt{y})$

$y = x^2$

$\pm\sqrt{y} = \sqrt{x^2}$

base $= 2\sqrt{y}$

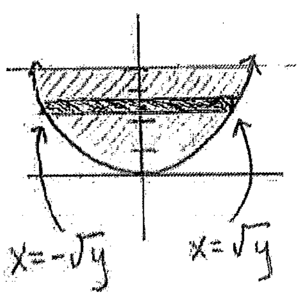
$V = \int_{y_1}^{y_2} [Area]dy$

$Area = \frac{1}{4}[2\sqrt{y}]^2 = \frac{1}{4}\cdot 4y = y$

$V = \int_0^4 y\,dy \rightarrow \frac{y^2}{2}\Big]_0^4 = \frac{4^2}{2} - 0 = 8$

$\boxed{V = 8 \text{ units}^3}$

**Example 3:** Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are semicircles whose base are perpendicular to the y-axis.




base (right-left): $\sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$

$A = \frac{\pi}{8}(base)^2$

$A = \frac{\pi}{8}(2\sqrt{y})^2 = \frac{\pi}{8}\cdot 4y$

$A = \frac{\pi}{2}y$

$V = \int_{y_1}^{y_2} Area\,dy$

$V = \frac{\pi}{2}\int_0^4 y\,dy \rightarrow \frac{y^2}{2}\Big]_0^4 = \frac{16}{2} = 8$

$\boxed{V = \frac{\pi}{2}(8) = 4\pi \text{ units}^3}$

**Example 4:** Let the region R be the area enclosed by the function $f(x) = \ln x$ and $g(x) = \frac{1}{2}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the $x$-axis is an isosceles right triangle with a leg in the region R, find the volume of the solid.

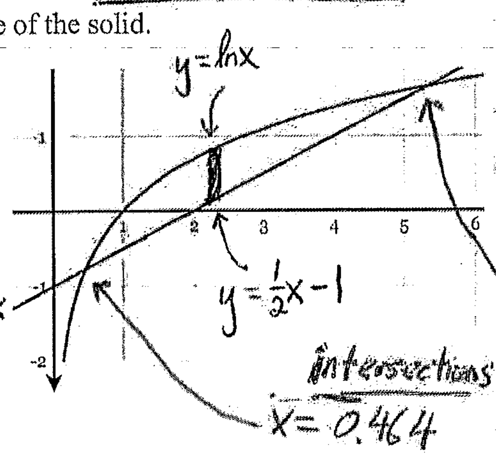
base (top-bottom): $\ln x - (\frac{1}{2}x - 1)$

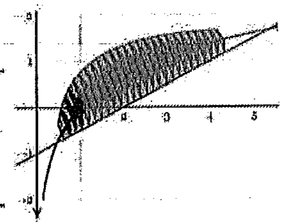
base $= \ln x - \frac{1}{2}x + 1$

$Area = \frac{1}{2}(base)^2 \rightarrow \frac{1}{2}(\ln x - \frac{1}{2}x + 1)^2$

$V = \int_{x_1}^{x_2} [Area]dx \rightarrow V = \int_{0.464}^{5.357} \frac{1}{2}(\ln x - \frac{1}{2}x + 1)^2 dx$

$\boxed{V = 0.613}$

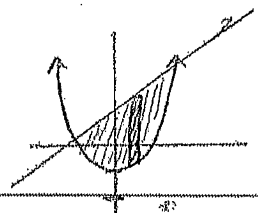


7.2c Homework p. 465-466 #61, 62, 63

*Volume of Known Cross Sections

61) $y = x+1$, $y = x^2-1$, perpendicular to x-axis



a) Squares

base $= x+1-(x^2-1)$

$= \cancel{x+1-x^2+1}$

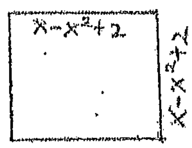
$= x - x^2 + 2$

*find bounds

$x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

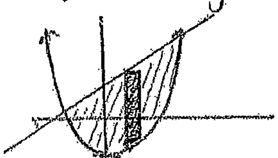
$(x-2)(x+1) = 0$ $\boxed{x = 2, -1}$




$V = \int_{-1}^{2} (\text{base})^2 dx = \int_{-1}^{2} [x - x^2 + 2]^2 dx$

$= \boxed{\frac{81}{10} \text{ units}^3}$

b) Rectangle of height 1



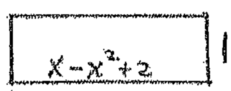
Area $=$ (base)(1)

base $= x+1-(x^2-1)$

$= x - x^2 + 2$

*bounds:

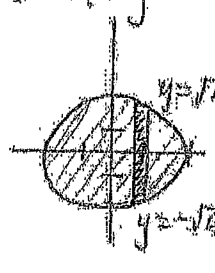
$x = -1, 2$




$V = \int_{-1}^{2} (x^2 - x + 2)(1)\, dx = \boxed{\frac{9}{2} \text{ units}^3}$

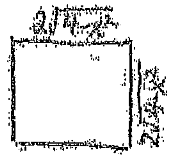
7.2c HW continued

62) $x^2+y^2=4$ perpendicular to x-axis




base $=\sqrt{4-x^2}-(-\sqrt{4-x^2})=2\sqrt{4-x^2}$

a) Squares

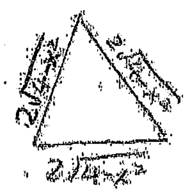



$V=\int_{-2}^{2}\left(2\sqrt{4-x^2}\right)^2dx$

$\boxed{V=\frac{128}{3}\text{ units}^3}$

b) Equilateral triangles

base $=2\sqrt{4-x^2}$



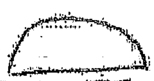

Area $=\frac{\sqrt{3}}{4}(\text{base})^2$

$V=\frac{\sqrt{3}}{4}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2dx$

$V=\frac{32\sqrt{3}}{3}\approx\boxed{18.475}$

c) Semicircles

base $=\sqrt{4-x^2}$



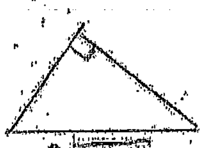

$A=\frac{\pi}{8}(\text{diameter})^2$

$V=\frac{\pi}{8}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2dx$

$\boxed{V=\frac{16}{3}\pi}$

d) Isosceles right triangles

base $=2\sqrt{4-x^2}$




hypotenuse $=2\sqrt{4-x^2}$

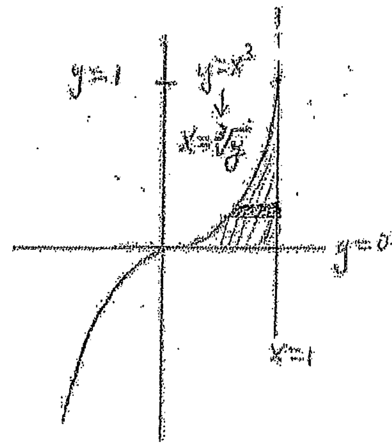
$A=\frac{1}{4}(\text{hypotenuse})^2$

$=\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$

$V=\int_{-2}^{2}\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$

$\boxed{V=\frac{32}{3}\text{ units}^3}$

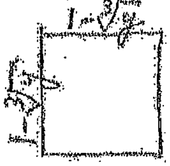
7.2c HW (continued)




63) graph bounded by $y=x^3$, $y=0$, $x=1$
(perpendicular to the y-axis)

base $= 1-\sqrt[3]{y}$ | *find bounds $\sqrt[3]{y}=1$, $y=1$

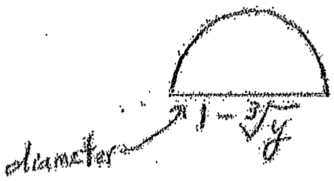
a) squares




$V=\int_0^1 (1-\sqrt[3]{y})^2\,dy$

$\boxed{V=\frac{1}{10}}$

b) semicircle

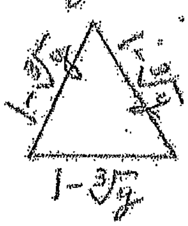



$A=\frac{\pi}{8}(\text{diameter})^2$

$V=\frac{\pi}{8}\int_0^1 [1-\sqrt[3]{y}]^2$

$V=\frac{\pi}{8}\left(\frac{1}{10}\right)=\boxed{\frac{\pi}{80}\text{ units}^3}$

c) Equilateral triangles



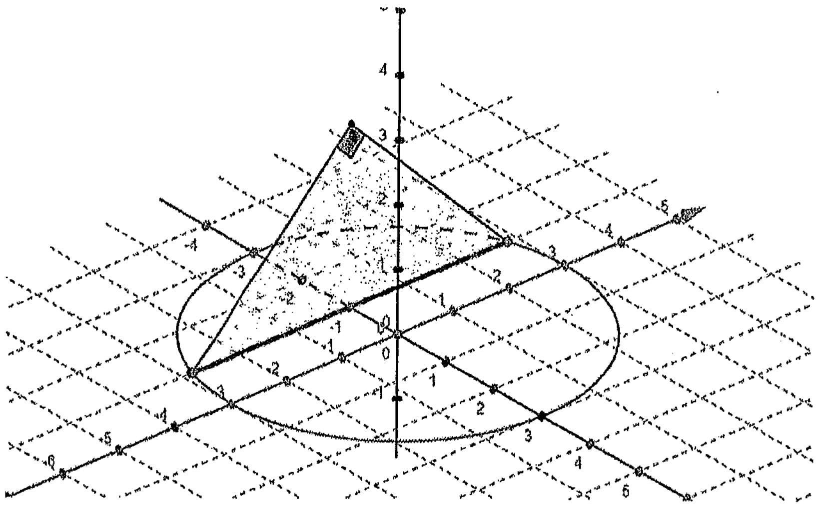

$A=\frac{\sqrt{3}}{4}(\text{base})^2$

$V=\frac{\sqrt{3}}{4}\int_0^1 [1-\sqrt[3]{y}]^2\,dy$

$V=\frac{\sqrt{3}}{4}\left(\frac{1}{10}\right)$

$=\boxed{\frac{\sqrt{3}}{40}\text{ units}^3}$

**Right Triangle with hypotenuse on the base**



**Right Triangle with leg on the base**