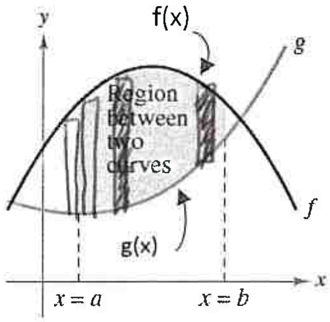
Key

AP Calculus Ch. 7.1 – Area Between Two Curves



Vertical Orientation: (vertical rectangles between graphs)

Right bound → x_2

$$\textbf{Area} = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

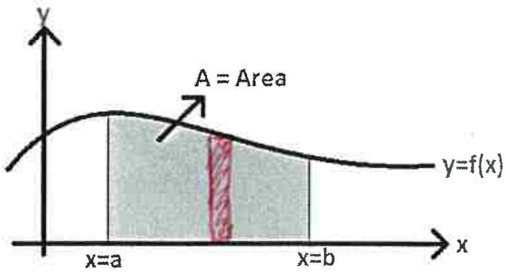
Left bound → x_1

Expressions in terms of **x**

(Equations in the form of " y = ___ ")

Example 1: Area = $\int_a^b f(x) - g(x)\, dx$

Example 2:



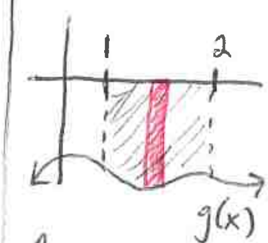
Top graph

bottom graph

$\int_a^b f(x) - 0\, dx$

Area = $\int_a^b f(x)\, dx$

Ex. 2b



Area = $\int_1^2 0 - g(x)\, dx$

$\boxed{\int_1^2 -g(x)\, dx}$

Example 3: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$

Steps:

i) Find bounds: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for x).

ii) Identify the **top and bottom** function

iii) Apply the Integral **Area Formula**.

* set equations equal:

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3,\ x = -1$

Top, bottom

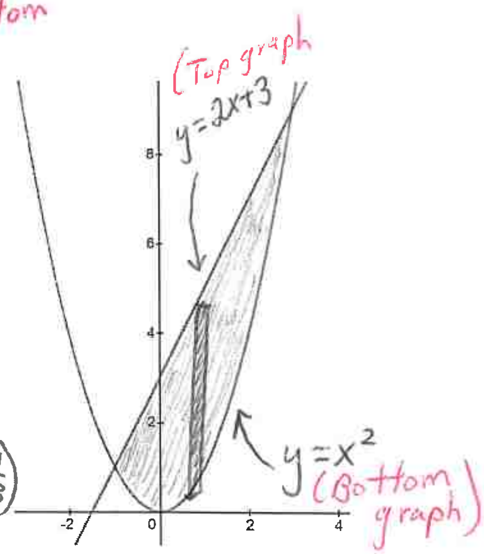
$\text{Area} = \int_{-1}^{3} (2x+3) - (x^2)\, dx$

$\int 2x + 3 - x^2\, dx$

$\left[\frac{\cancel{2}x^2}{\cancel{2}} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

$3^2 + 3(3) - \frac{3^3}{3} - \left((-1)^2 + 3(-1) + \frac{1}{3}\right)$

$\boxed{\text{Area} = \frac{32}{3}}$



Horizontal Orientation: (horizontal rectangles between graphs)

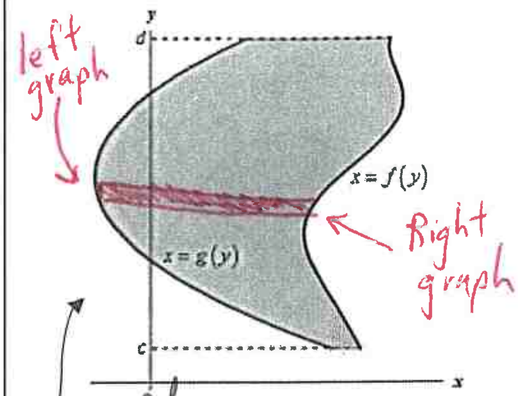
Upper bound

$$\textbf{Area} = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)\, dy$$

Lower bound

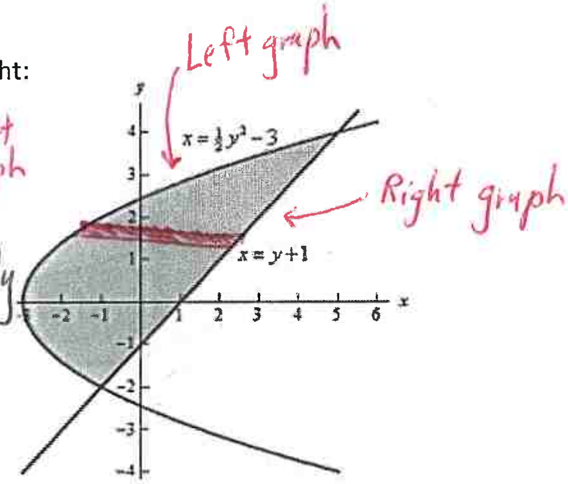
Expressions in terms of y

(Equations in the form of " x = ___ ")



Example 3: Area = $\int_c^d f(y) - g(y)\, dy$

Example 4: Find area of the region bounded by the equations on right:



Steps:

i) Find bounds: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for y).

ii) Identify the **right and left** function

iii) Apply the Integral **Area Formula**

* find bounds (intersection)

$\frac{1}{2}y^2 - 3 = y + 1$

$2\left(\frac{1}{2}y^2 - y - 4 = 0\right)$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = 4,\ y = -2$

Right graph Left graph

$\text{Area} = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$

$\int y + 1 - \frac{1}{2}y^2 + 3\, dy$

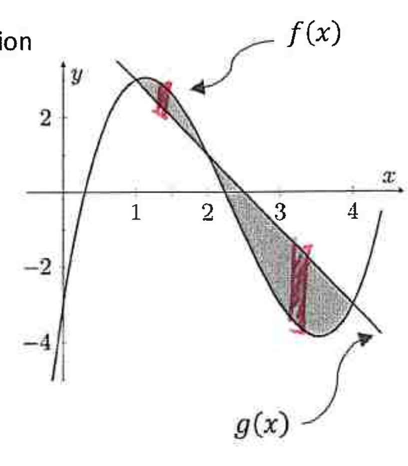
$\int y - \frac{1}{2}y^2 + 4\, dy$

$\left.\frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^3}{3} + 4y\right]_{-2}^{4} = \frac{4^2}{2} - \frac{4^3}{6} + 4(4) - \left(\frac{-2^2}{2} - \frac{(-2)^3}{6} - 8\right) = 18$

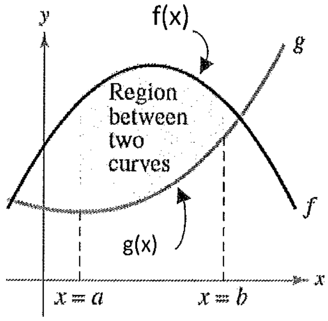
$\boxed{\text{Area} = 18}$

Example 5: Represent the area of shaded region to the right using integral notation

$\boxed{\text{Area} = \int_1^2 f(x) - g(x)\, dx + \int_2^4 g(x) - f(x)\, dx}$



AP Calculus Ch. 7.1 – Area Between Two Curves



Vertical Orientation: (vertical rectangles between graphs)

Right bound

$$\boldsymbol{Area} = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\ dx$$

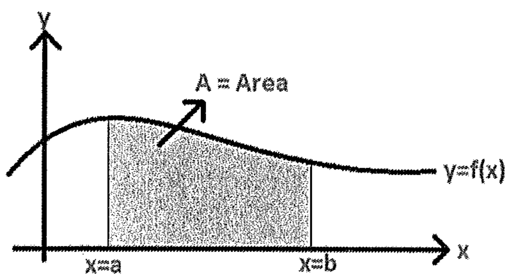
Left bound

Expressions in terms of **x**

(Equations in the form of " y = ___ ")

Example 1: Area = ______________________

Example 2:



Area = ______________________

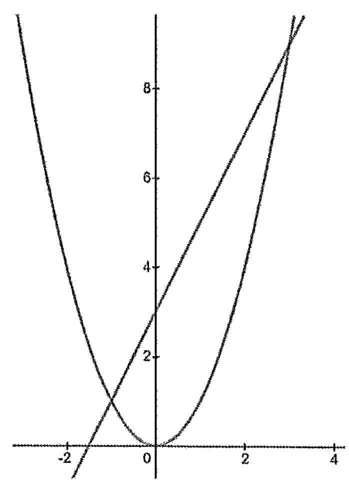
Example 3: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$

Steps:

i) Find bounds: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for x).

ii) Identify the **top and bottom** function

iii) Apply the Integral **Area Formula.**



Horizontal Orientation: (horizontal rectangles between graphs)

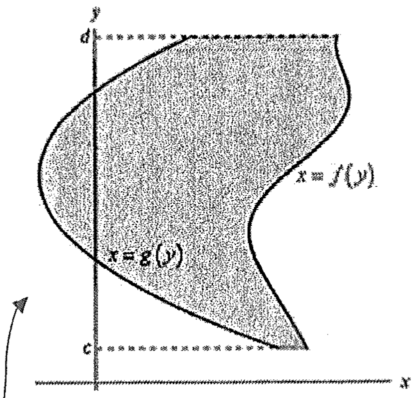
Upper bound

$$\textbf{Area} = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)\, dy$$

Lower bound

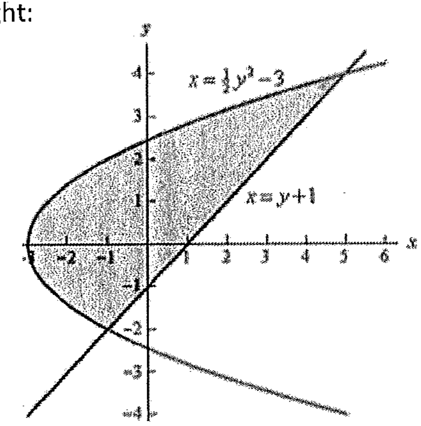
Expressions in terms of y

(Equations in the form of " x = ___")



Example 3: Area = ________________________

Example 4: Find area of the region bounded by the equations on right:



Steps:

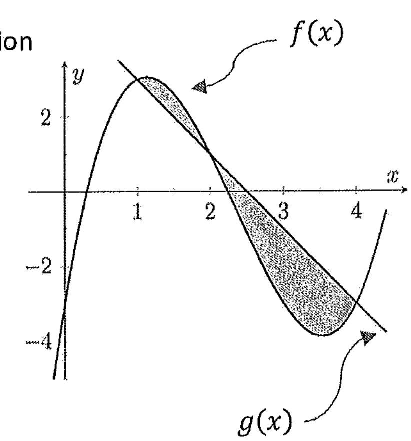
i) Find bounds: Find the point of intersection between the 2 graphs

(by setting equations equal, & solving for y).

ii) Identify the **right and left** function

iii) Apply the Integral **Area Formula**

Example 5: Represent the area of shaded region to the right using integral notation

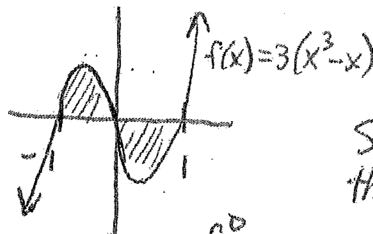


7.1 Homework p. 452-453 # 1, 3, 5, 17-35 odd, 43, 47, 71

*Area between curves

5) $f(x) = 3(x^3 - x)$ Set up definite integral giving area of region

$g(x) = 0$

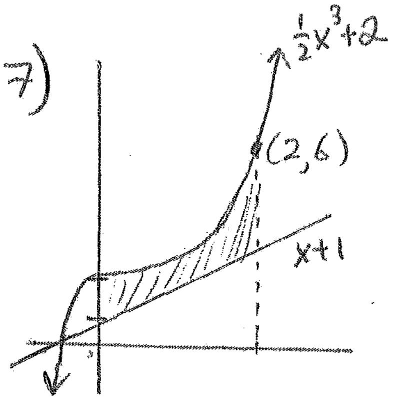


Since f(x) is an odd function, the area regions are equal to each other.

$$A = \int_{-1}^{0} 3(x^3-x)\,dx + \int_{-1}^{0} 3(x^3-x)\,dx = 2\int_{-1}^{0} 3(x^3-x)\,dx$$

$$= \boxed{6\int_{-1}^{0} (x^3-x)\,dx}$$

17)



right bound

$$A = \int_0^2 \Big[\underbrace{\left(\tfrac{1}{2}x^3+2\right)}_{\text{top function}} - \underbrace{(x+1)}_{\text{bottom function}}\Big]\,dx = \int \tfrac{1}{2}x^3 + 2 - x - 1\,dx$$

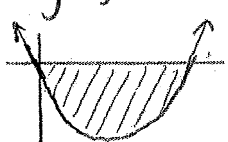
left bound

$$= \int \tfrac{1}{2}x^3 + 1 - x\,dx = \tfrac{1}{2}\left(\frac{x^4}{4}\right) + x - \frac{x^2}{2}\Big]_0^2 = \frac{2^4}{8} + 2 - \frac{4}{2} - (0+0-0)$$

$$= 2 + 2 - 2 = \boxed{2}$$

19) $f(x) = x^2 - 4x$

$g(x) = 0$



*set equations equal to each other to find left and right bounds

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0,\ x = 4$

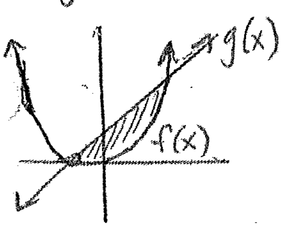
$$\int_0^4 \underbrace{0}_{\text{top}} - \underbrace{(x^2-4x)}_{\text{bottom}}\,dx = \int_0^4 -x^2 + 4x\,dx = -\frac{x^3}{3} + \frac{4x^2}{2}\Big]_0^4$$

$$= -\frac{4^3}{3} + 2(4)^2 = \frac{-64}{3} + 32$$

$$= \boxed{\frac{32}{3}}$$

21) $f(x) = x^2 + 2x + 1$

$g(x) = 3x + 3$



*find left/right bounds:

$x^2 + 2x + 1 = 3x + 3$

$x^2 - 1x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$$\int_{-1}^{2} \underbrace{3x+3}_{\text{top}} - \underbrace{(x^2+2x+1)}_{\text{bottom}}\,dx$$

$3x + 3 - x^2 - 2x - 1$

$-x^2 + x + 2$

$$\int_{-1}^{2} -x^2 + x + 2\,dx = \frac{-x^3}{3} + \frac{x^2}{2} + 2x\Big]_{-1}^{2}$$

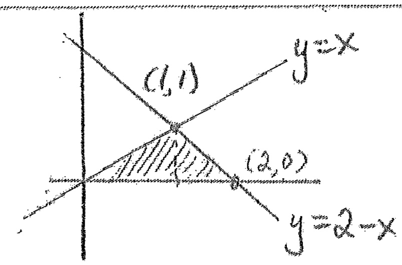
$$\frac{-8}{3} + \frac{4}{2} + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$\frac{-8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2$$

$$= \boxed{\frac{9}{2}}$$

7.1 HW (continued)

23) $y=x$, $y=2-x$, $y=0$



$y=x$ and $y=2-x$ intersect at $x=1$

Method 1

$$\int_0^1 (\underbrace{x}_{top}-\underbrace{0}_{bottom})dx + \int_1^2 (\underbrace{2-x}_{top}-\underbrace{0}_{bottom})dx$$

$$\frac{x^2}{2}\Big]_0^1 = \frac{1}{2} \qquad 2x-\frac{x^2}{2}\Big]_1^2$$

$$=4-2-(2-\tfrac{1}{2})$$

$$=\frac{1}{2}+\cancel{4}-\cancel{2}-\cancel{2}+\frac{1}{2} = \boxed{1}$$

Method 2

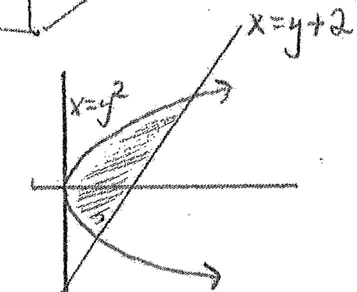
$$\int_{y_1}^{y_2}(\text{Right}-\text{Left})dy \qquad \text{Left: } x=y \qquad \text{Right: } x=2-y$$

$$\int_{0\ (\text{Lower bound})}^{1\ (\text{upper bound})} \underbrace{2-y}_{\text{Right}}-\underbrace{y}_{\text{Left}}\,dy = \int 2-2y = 2y-\frac{\cancel{2}y^2}{\cancel{2}}\Big]_0^1$$

$$=2-1-(0-0)=\boxed{1}$$

27) $f(y)=y^2 \rightarrow x=y^2$

$g(y)=y+2 \qquad x=y+2$



$$A=\int_{y_1}^{y_2}\text{Right}-\text{Left}\,dy$$

*find lower/upper bounds

$y^2=y+2 \qquad y^2-y+2=0$

$(y-2)(y+1)=0$

$y=2,-1$

$$A=\int_{-1}^{2}\underbrace{y+2}_{\text{Right}}-\underbrace{y^2}_{\text{Left}}dy = \frac{y^2}{2}+2y-\frac{y^3}{3}\Big]_{-1}^{2}$$

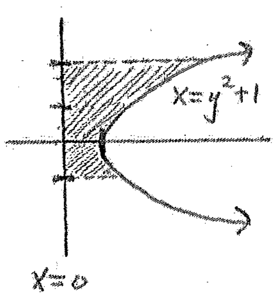
$$=\frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right) = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}$$

$$=2+4-\frac{9}{3}-\frac{1}{2}+2=\boxed{\frac{9}{2}}$$

29) $f(y)=y^2+1$

$g(y)=0$, $y=-1$, $y=2 \rightarrow x=y^2+1 \quad y=-1$

$x=0 \quad y=2$



$$\int_{-1}^{2}\overbrace{y^2+1}^{\text{Right}}-\overbrace{0}^{\text{Left}}dy$$

$$\frac{y^3}{3}+y\Big]_{-1}^{2} = \frac{8}{3}+2-\left(\frac{-1}{3}-1\right)$$

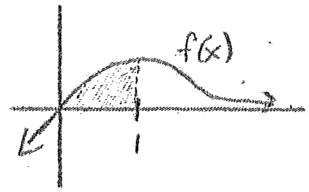
$$=\frac{8}{3}+2+\frac{1}{3}+1 = 3+2+1=\boxed{6}$$

7.1 HW (continued)

47) $f(x) = xe^{-x^2}$

$y = 0$

$0 \le x \le 1$



$$A = \int_0^1 \underbrace{xe^{-x^2}}_{top} - \underbrace{0}_{bottom}\,dx$$

$$\int_0^1 xe^{-x^2}\,dx \qquad u = -x^2 \qquad \frac{du}{dx} = -2x \qquad dx = \frac{du}{-2x}$$

$$\int \cancel{x} \cdot e^u \cdot \frac{du}{-2\cancel{x}} \qquad \frac{-1}{2}\int e^u\,du$$

if $x=0$, $u = -(0)^2 = 0$

if $x=1$, $u = -(1)^2 = -1$

$$\frac{-1}{2}e^u\Big]_0^{-1} = \frac{-1}{2}e^{-1} - \left(\frac{-1}{2}e^0\right)$$

$$= \frac{-1}{2e} + \frac{1}{2} \approx \boxed{0.316}$$

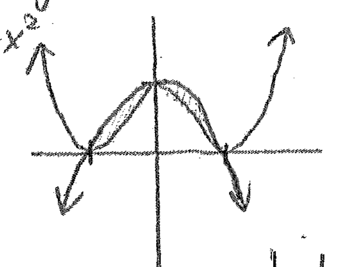
71) The graphs $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ can be found by single integral.

$x^4 - 2x^2 + 1 = 1 - x^2$

$x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$x = 0, 1, -1$



Since $1 - x^2 \ge x^4 - 2x^2 + 1$, $y = 1 - x^2$ will always be the top curve. There is therefore no need to split into 2 integrals.

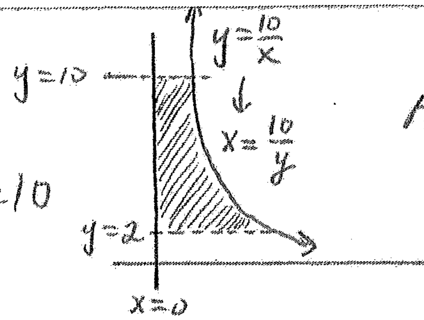
$$A = \int_{-1}^1 1 - x^2 - (x^4 - 2x^2 + 1)\,dx$$

$1 - x^2 - x^4 + 2x^2 - 1 = -x^4 + x^2$

$$A = \int_{-1}^1 x^2 - x^4\,dx = \frac{x^3}{3} - \frac{x^5}{5}\Big]_{-1}^1 = \frac{1}{3} - \frac{1}{5} - \left(\frac{-1}{3} + \frac{1}{5}\right)$$

$$= \frac{2}{3} - \frac{2}{5} = \boxed{\frac{4}{15}}$$

7.1 HW (continued)

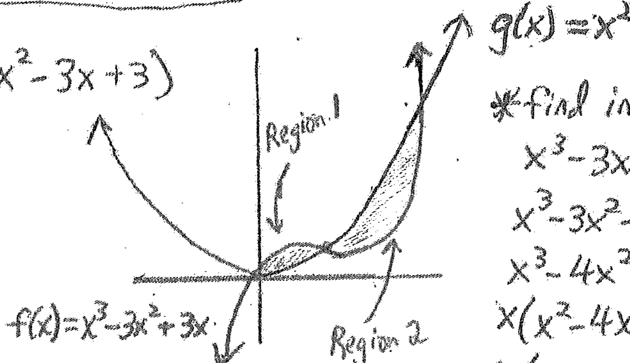


$$A=\int_{y_1}^{y_2}(\text{Right}-\text{Left})\,dy$$

31) $f(x)=\frac{10}{x}$, $x=0$, $y=2$, $y=10$

$$\int_2^{10}\overbrace{\frac{10}{y}}^{\text{Right}}-\overbrace{0}^{\text{Left}}\,dy \quad 10\ln|y|\Big]_2^{10} = 10\ln(10)-10\ln(2) = 10\ln\left(\frac{10}{2}\right)=10\ln 5 \approx \boxed{16.0944}$$

33) $f(x)=x(x^2-3x+3)$
$g(x)=x^2$



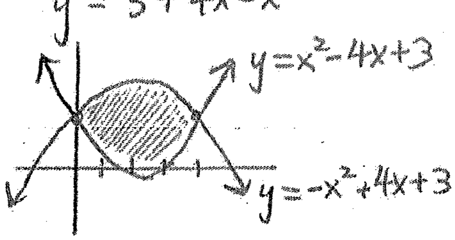
*find intersections to determine bounds:

$x^3-3x^2+3x=x^2$
$x^3-3x^2-x^2+3x=0$
$x^3-4x^2+3x=0$
$x(x^2-4x+3)=0$
$x(x-3)(x-1)=0$

$\underline{x=0,\ x=3,\ x=1}$

$$A=\int_0^1 \underbrace{x^3-3x^2+3x}_{\text{top}}-\underbrace{(x^2)}_{\text{bottom}}\,dx+\int_1^3 \underbrace{x^2}_{\text{top}}-\underbrace{(x^3-3x^2+3x)}_{\text{bottom}}\,dx = \boxed{\frac{37}{12}}$$

35) $y=x^2-4x+3$
$y=3+4x-x^2$



*find points of intersection:

$x^2-4x+3=-x^2+4x+3$
$2x^2-8x=0$
$2x(x-4)=0$
$\underline{x=0,\ x=4}$

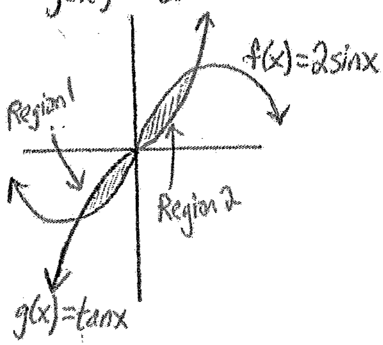
$$A=\int_0^4 -x^2+4x+3-(x^2-4x+3)\,dx$$

$$\int -2x^2+8x\,dx = -\frac{2x^3}{3}+\frac{8x^2}{2}\Big]_0^4$$

$$-\frac{2}{3}x^3+4x^2\Big]_0^4 = -\frac{128}{3}+64-(0+0)$$

$$=\boxed{\frac{64}{3}}$$

43) $f(x)=2\sin x$ $\quad -\frac{\pi}{3}\le x\le\frac{\pi}{3}$
$g(x)=\tan x$



*Region 1 = Region 2

$$A=2\cdot\int_0^{\pi/3} 2\sin x-\tan x\,dx$$

$$=2\cdot\left(-2\cos x+\ln|\cos x|\right)\Big]_0^{\pi/3}$$

$=2\left(-2\cos(\pi/3)+\ln|\cos(\pi/3)|\right)-2(-2(1)+0)$
$=-4\left(\frac{1}{2}\right)+2\ln\left|\frac{1}{2}\right|+4$
$=-2+4+2\ln(0.5)$
$=2+2\ln(0.5)\approx\boxed{0.614}$