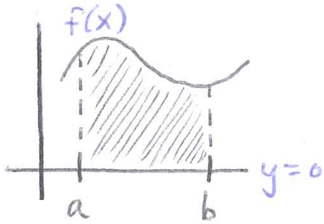
7.1 Notes — Area between 2 curves (graphs)

$$\text{Area} = \int_{x_1}^{x_2} (\text{Top function} - \text{Bottom function})\,dx$$

right bound $\rightarrow x_2$

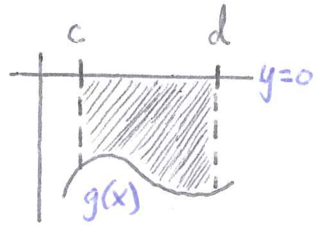
left bound $\rightarrow x_1$

All variables/values in terms of x



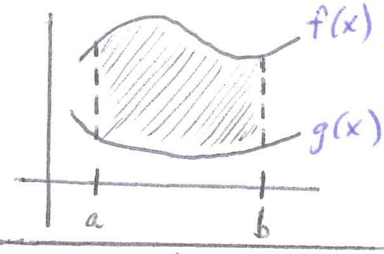
$$\text{Area} = \int_a^b f(x) - 0\,dx$$

$$= \boxed{\int_a^b f(x)\,dx}$$



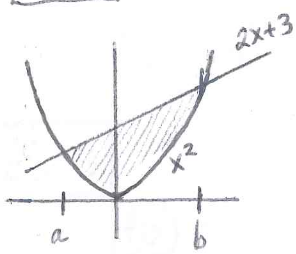
$$\text{Area} = \int_c^d 0 - g(x)\,dx$$

$$= \boxed{-\int_c^d g(x)\,dx}$$



$$\boxed{\text{Area} = \int_a^b f(x) - g(x)\,dx}$$

Ex. 1 Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$



Steps:

1) Find points of intersection (to locate left and right bounds for region)
2) Identify top and bottom function
3) Plug into Area formula and solve.

*To find intersection, set equations equal to each other.

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = -1, 3$

$$\text{Area} = \int_{-1}^{3} \underbrace{2x+3}_{\text{top}} - \underbrace{(x^2)}_{\text{bottom}}\,dx$$

$$\int_{-1}^{3} 2x+3-x^2\,dx$$

$$= \frac{2x^2}{2} + 3x - \frac{x^3}{3}\Bigg]_{-1}^{3}$$

$$= (3)^2 + 3(3) - \frac{3^3}{3} - \left(1 - 3 + \frac{1}{3}\right)$$

$$= 9 + 9 - 9 - 1 + 3 - \frac{1}{3}$$

$$= 11 - \frac{1}{3} = \boxed{\frac{32}{3}}$$

$$\boxed{\text{Area} = \frac{32}{3}}$$

7.1 Notes (continued)

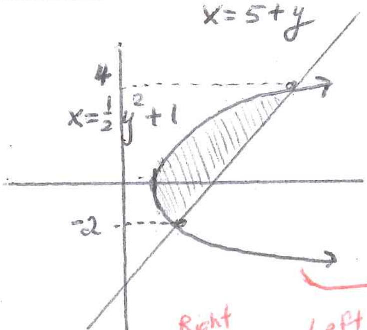
$$\text{Area}_{(\text{sideways})} = \int_{y_1}^{y_2} [\text{Right graph} - \text{Left graph}]\, dy$$

upper bound (y-value) → y_2

lower bound (y-value) → y_1

All variables/values in terms of y

Ex. 2 Find area of region bounded by $y = \pm\sqrt{2x-2}$ and $y = x-5$



$y^2 = 2x-2$

$y^2 + 2 = 2x$

$\frac{1}{2}(y^2+2) = x$

$x = \frac{1}{2}y^2 + 1$ (left graph)

$y = x-5$

$x = 5+y$ (right graph)

Find points of intersection:

$\frac{1}{2}y^2 + 1 = 5 + y$

$2\left(\frac{1}{2}y^2 - y - 4 = 0\right)$

$y^2 - 2y - 8 = 0$

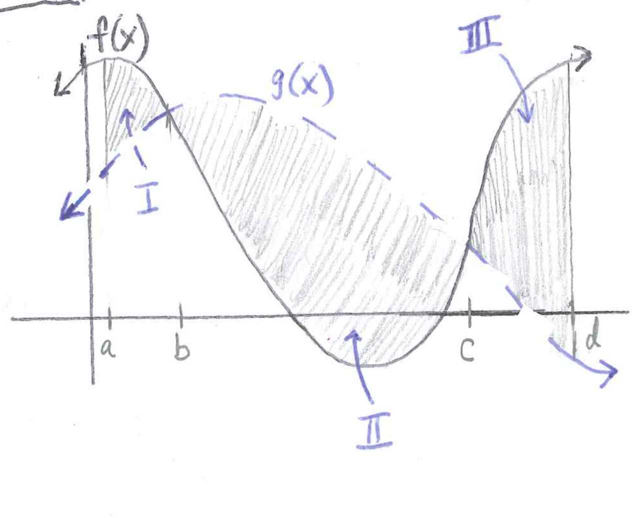
$(y-4)(y+2) = 0$

$y = 4, -2$ (bounds)

$$= \int_{-2}^{4} \underbrace{(5+y)}_{\text{Right graph}} - \underbrace{\left(\tfrac{1}{2}y^2+1\right)}_{\text{Left graph}} dy$$

$$= \int_{-2}^{4} -\tfrac{1}{2}y^2 + y + 4\, dy = -\tfrac{1}{2}\left(\tfrac{y^3}{3}\right) + \tfrac{y^2}{2} + 4y \Big]_{-2}^{4} = \left(\tfrac{-64}{6} + \tfrac{16}{2} + 16\right) - \left(\tfrac{8}{6} + 2 - 8\right) = \boxed{18}$$

Ex. 3



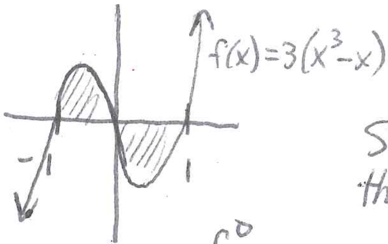
* To find Area, write separate integrals due to alternating top and bottom functions.

$$\text{Area} = \int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx + \int_c^d f(x) - g(x)\,dx$$

7.1 Homework p. 452-453 # 1, 3, 5, 17-35 odd, 43, 47, 71

*Area between curves

5) $f(x) = 3(x^3 - x)$ Set up definite integral giving area of region
$g(x) = 0$

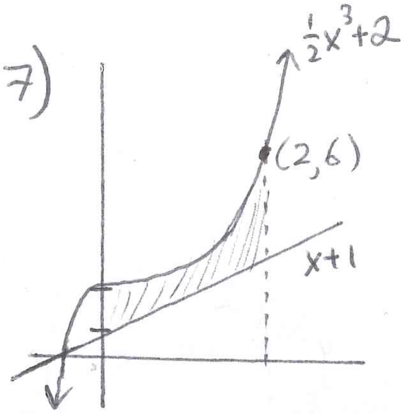


Since $f(x)$ is an odd function, the area regions are equal to each other.

$$A = \int_{-1}^{0} 3(x^3-x)\,dx + \int_{-1}^{0} 3(x^3-x)\,dx = 2\int_{-1}^{0} 3(x^3-x)\,dx$$

$$= \boxed{6\int_{-1}^{0}(x^3-x)\,dx}$$

17)



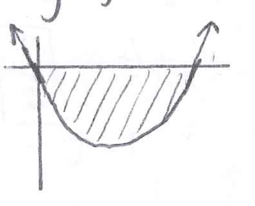
$$A = \int_{0}^{2} \Big[\underbrace{\left(\tfrac{1}{2}x^3+2\right)}_{\text{top function}} - \underbrace{(x+1)}_{\text{bottom function}}\Big]\,dx = \int \tfrac{1}{2}x^3 + 2 - x - 1\,dx$$

right bound (2), left bound (0)

$$= \int \tfrac{1}{2}x^3 + 1 - x\,dx = \tfrac{1}{2}\left(\tfrac{x^4}{4}\right) + x - \tfrac{x^2}{2}\Big]_0^2 = \tfrac{2^4}{8} + 2 - \tfrac{4}{2} - (0+0-0)$$

$$= 2 + 2 - 2 = \boxed{2}$$

19) $f(x) = x^2 - 4x$
$g(x) = 0$



* set equations equal to each other to find left and right bounds

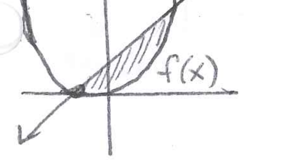
$x^2 - 4x = 0$
$x(x-4) = 0$
$x = 0,\ x = 4$

$$\int_0^4 \underbrace{0}_{\text{top}} - \underbrace{(x^2-4x)}_{\text{bottom}}\,dx = \int_0^4 -x^2+4x\,dx = -\frac{x^3}{3} + \frac{4x^2}{2}\Big]_0^4$$

$$= -\frac{4^3}{3} + 2(4)^2 = \frac{-64}{3} + 32$$

$$= \boxed{\frac{32}{3}}$$

21) $f(x) = x^2 + 2x + 1$
$g(x) = 3x + 3$



*find left/right bounds:

$x^2 + 2x + 1 = 3x + 3$
$x^2 - 1x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2,\ x = -1$

$$\int_{-1}^{2} \underbrace{3x+3}_{\text{top}} - \underbrace{(x^2+2x+1)}_{\text{bottom}}\,dx$$

$3x + 3 - x^2 - 2x - 1$
$-x^2 + x + 2$

$$\int_{-1}^{2} -x^2 + x + 2\,dx = \frac{-x^3}{3} + \frac{x^2}{2} + 2x\Big]_{-1}^{2}$$

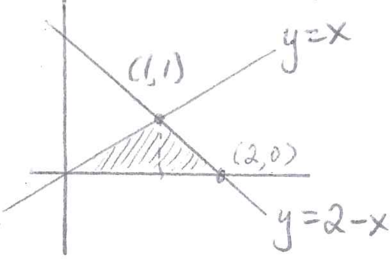
$$\frac{-8}{3} + \frac{4}{2} + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$-\frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2$$

$$= \boxed{\frac{9}{2}}$$

7.1 HW (continued)

23) $y=x,\ y=2-x,\ y=0$



$y=x$ and $y=2-x$ intersect at $x=1$

Method 1

$$\int_0^1 (\underbrace{x}_{\text{top}} - \underbrace{0}_{\text{bottom}})dx + \int_1^2 (\underbrace{2-x}_{\text{top}} - \underbrace{0}_{\text{bottom}})dx$$

$$\left.\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \qquad \left. 2x-\frac{x^2}{2}\right]_1^2$$

$$=4-2-(2-\tfrac{1}{2})$$

$$=\frac{1}{2}+\cancel{4}-\cancel{2}-\cancel{2}+\frac{1}{2} = \boxed{1}$$

Method 2

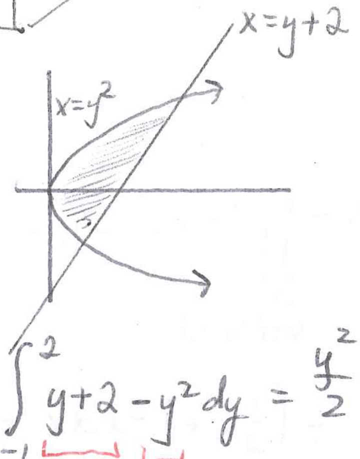
$$\int_{y_1}^{y_2} (\text{Right}-\text{Left})\,dy \qquad \text{Left: } x=y \qquad \text{Right: } x=2-y$$

$$\int_{\underset{\text{Lower bound}}{0}}^{\overset{\text{upper bound}}{1}} \underbrace{2-y}_{\text{Right}} - \underbrace{y}_{\text{Left}}\, dy = \int 2-2y = \left. 2y - \frac{\cancel{2}y^2}{\cancel{2}}\right]_0^1$$

$$=2-1-(0-0) = \boxed{1}$$

27) $f(y)=y^2 \rightarrow x=y^2$

$g(y)=y+2 \qquad x=y+2$



$$A=\int_{y_1}^{y_2} \text{Right} - \text{Left}\ dy$$

*find lower/upper bounds

$y^2=y+2 \qquad y^2-y+2=0$

$(y-2)(y+1)=0$

$y=2,-1$

$$A=\int_{-1}^{2} \underbrace{y+2}_{\text{Right}} - \underbrace{y^2}_{\text{Left}} dy = \left.\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}$$

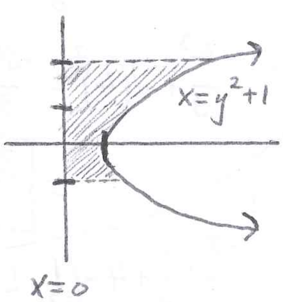
$$=\frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right) = 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}$$

$$=2+4-\frac{9}{3}-\frac{1}{2}+2 = \boxed{\frac{9}{2}}$$

29) $f(y)=y^2+1$

$g(y)=0,\ y=-1,\ y=2 \rightarrow x=y^2+1 \quad y=-1$

$x=0 \quad y=2$



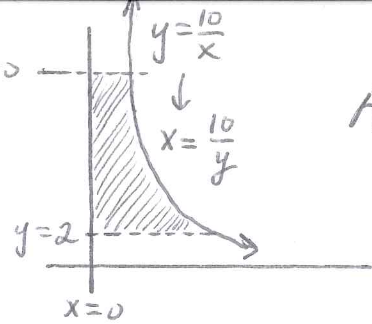
$$\int_{-1}^{2} \overbrace{y^2+1}^{\text{Right}} - \overbrace{0}^{\text{Left}}\, dy$$

$$\left.\frac{y^3}{3}+y\right]_{-1}^{2} = \frac{8}{3}+2-\left(\frac{-1}{3}-1\right)$$

$$=\frac{8}{3}+2+\frac{1}{3}+1 = 3+2+1 = \boxed{6}$$

7.1 HW (continued)

31) $f(x)=\frac{10}{x}$, $x=0$, $y=2$, $y=10$

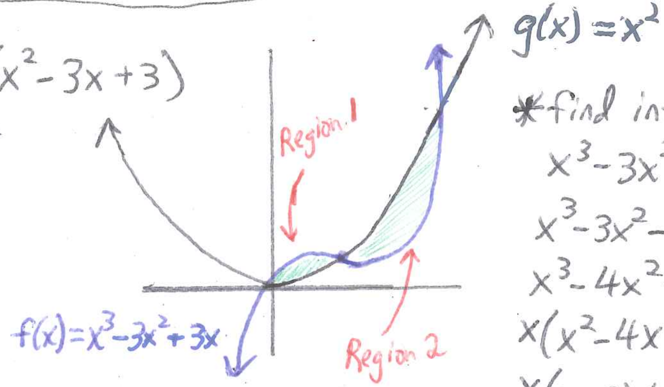


$$A=\int_{y_1}^{y_2}(\text{Right}-\text{Left})\,dy$$

$$\int_2^{10}\underbrace{\frac{10}{y}}_{\text{Right}}-\underbrace{0}_{\text{Left}}\,dy \quad 10\ln|y|\Big]_2^{10} = 10\ln(10)-10\ln(2) = 10\ln\left(\frac{10}{2}\right) = 10\ln 5 \approx \boxed{16.0944}$$

33) $f(x)=x(x^2-3x+3)$

$g(x)=x^2$



*find intersections to determine bounds:

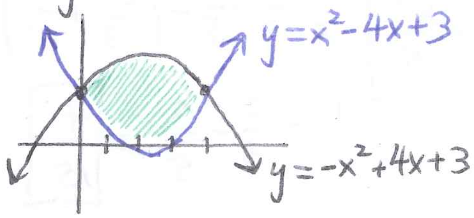
$$x^3-3x^2+3x=x^2$$
$$x^3-3x^2-x^2+3x=0$$
$$x^3-4x^2+3x=0$$
$$x(x^2-4x+3)=0$$
$$x(x-3)(x-1)=0$$

$x=0, x=3, x=1$

$$A=\int_0^1 \underbrace{x^3-3x^2+3x}_{\text{top}}-\underbrace{(x^2)}_{\text{bottom}}\,dx + \int_1^3 \underbrace{x^2}_{\text{top}}-\underbrace{(x^3-3x^2+3x)}_{\text{bottom}}\,dx = \boxed{\frac{37}{12}}$$

35) $y=x^2-4x+3$

$y=3+4x-x^2$



*find points of intersection:

$$x^2-4x+3=-x^2+4x+3$$
$$2x^2-8x=0$$
$$2x(x-4)=0$$
$$x=0,\ x=4$$

$$A=\int_0^4 -x^2+4x+3-(x^2-4x+3)\,dx$$

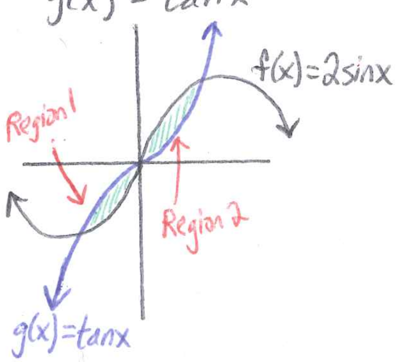
$$\int -2x^2+8x\,dx = -\frac{2x^3}{3}+\frac{8x^2}{2}\Big]_0^4$$

$$-\frac{2}{3}x^3+4x^2\Big]_0^4 = -\frac{128}{3}+64-(0+0)$$

$$=\boxed{\frac{64}{3}}$$

43) $f(x)=2\sin x$ $\quad -\frac{\pi}{3}\le x\le\frac{\pi}{3}$

$g(x)=\tan x$



*Region 1 = Region 2

$$A=2\cdot\int_0^{\pi/3} 2\sin x-\tan x\,dx$$

$$=2\cdot\left(-2\cos x+\ln|\cos x|\right)\Big]_0^{\pi/3}$$

$$=2\left(-2\cos(\tfrac{\pi}{3})+\ln|\cos(\tfrac{\pi}{3})|\right)-2(-2(1)+0)$$

$$=-4\left(\tfrac{1}{2}\right)+2\ln\left|\tfrac{1}{2}\right|+4$$

$$=-2+4+2\ln(0.5)$$

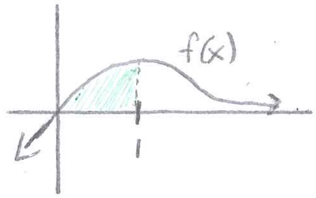
$$=2+2\ln(0.5)\approx\boxed{0.614}$$

7.1 HW (continued)

47) $f(x) = xe^{-x^2}$

$y = 0$

$0 \le x \le 1$



$$A = \int_0^1 \underbrace{xe^{-x^2}}_{top} - \underbrace{0}_{bottom}\, dx$$

$$\int_0^1 xe^{-x^2}\,dx \qquad u = -x^2 \qquad \frac{du}{dx} = -2x \qquad dx = \frac{du}{-2x}$$

$$\int \cancel{x} \cdot e^u \cdot \frac{du}{-2\cancel{x}}$$

$$-\frac{1}{2}\int e^u\,du$$

if $x = 0$, $u = -(0)^2 = 0$

if $x = 1$, $u = -(1)^2 = -1$

$$-\frac{1}{2}e^u\Big]_0^{-1} = -\frac{1}{2}e^{-1} - \left(-\frac{1}{2}e^0\right)$$

$$= -\frac{1}{2e} + \frac{1}{2} \approx \boxed{0.316}$$

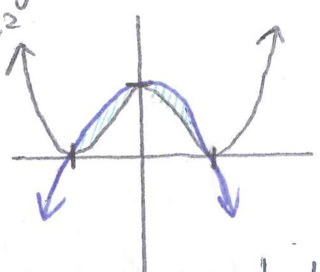
71) The graphs $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ can be found by single integral.

$x^4 - 2x^2 + 1 = 1 - x^2$

$x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$x = 0, 1, -1$



Since $1 - x^2 \ge x^4 - 2x^2 + 1$, $y = 1 - x^2$ will always be the top curve. There is therefore no need to split into 2 integrals.

$$A = \int_{-1}^{1} 1 - x^2 - (x^4 - 2x^2 + 1)\,dx$$

$1 - x^2 - x^4 + 2x^2 - 1 = -x^4 + x^2$

$$A = \int_{-1}^{1} x^2 - x^4\,dx = \frac{x^3}{3} - \frac{x^5}{5}\Big]_{-1}^{1} = \frac{1}{3} - \frac{1}{5} - \left(-\frac{1}{3} + \frac{1}{5}\right)$$

$$= \frac{2}{3} - \frac{2}{5} = \boxed{\frac{4}{15}}$$