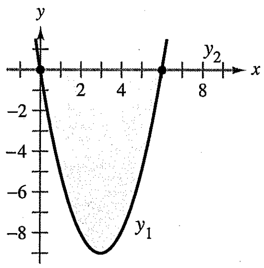
7.1 Exercises

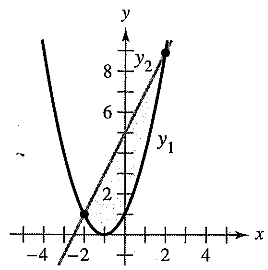
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Writing a Definite Integral **In Exercises 1–6, set up the definite integral that gives the area of the region.**

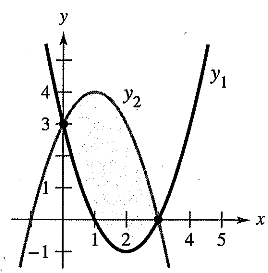
1. $y_1 = x^2 - 6x$
   $y_2 = 0$



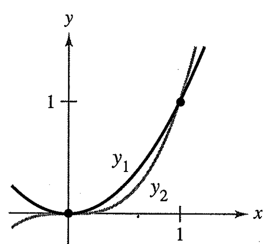
2. $y_1 = x^2 + 2x + 1$
   $y_2 = 2x + 5$



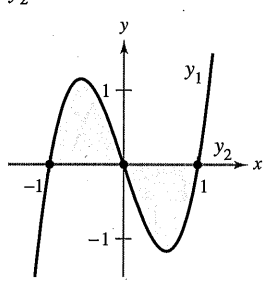
3. $y_1 = x^2 - 4x + 3$
   $y_2 = -x^2 + 2x + 3$



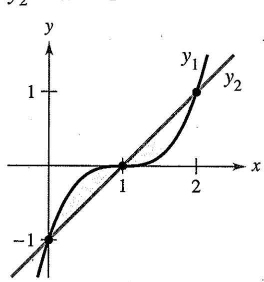
4. $y_1 = x^2$
   $y_2 = x^3$



5. $y_1 = 3(x^3 - x)$
   $y_2 = 0$



6. $y_1 = (x - 1)^3$
   $y_2 = x - 1$



Finding a Region **In Exercises 7–12, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.**

7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$

8. $\int_{-1}^1 [(2 - x^2) - x^2]\, dx$

9. $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$

10. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

11. $\int_{-2}^1 [(2 - y) - y^2]\, dy$

12. $\int_0^4 \left(2\sqrt{y} - y\right) dy$

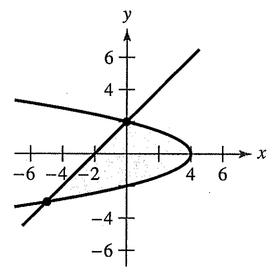
Think About It **In Exercises 13 and 14, determine which value best approximates the area of the region bounded by the graphs of f and g. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

13. $f(x) = x + 1, \quad g(x) = (x - 1)^2$
    (a) -2 (b) 2 (c) 10 (d) 4 (e) 8

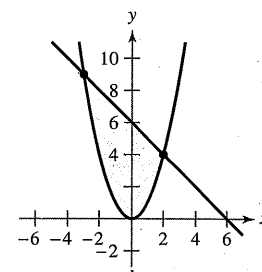
14. $f(x) = 2 - \frac{1}{2}x, \quad g(x) = 2 - \sqrt{x}$
    (a) 1 (b) 6 (c) -3 (d) 3 (e) 4

Comparing Methods **In Exercises 15 and 16, find the area of the region by integrating (a) with respect to x and (b) with respect to y. (c) Compare your results. Which method is simpler? In general, will this method always be simpler than the other one? Why or why not?**

15. $x = 4 - y^2$
    $x = y - 2$



16. $y = x^2$
    $y = 6 - x$



Finding the Area of a Region **In Exercises 17–30, sketch the region bounded by the graphs of the equations and find the area of the region.**

17. $y = x^2 - 1, \quad y = -x + 2, \quad x = 0, \quad x = 1$

18. $y = -x^3 + 2, \quad y = x - 3, \quad x = -1, \quad x = 1$

19. $f(x) = x^2 + 2x, \quad g(x) = x + 2$

20. $y = -x^2 + 3x + 1, \quad y = -x + 1$

21. $y = x, \quad y = 2 - x, \quad y = 0$

22. $y = \frac{4}{x^3}, \quad y = 0, \quad x = 1, \quad x = 4$

23. $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$

24. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x - 1$

25. $f(y) = y^2, \quad g(y) = y + 2$

26. $f(y) = y(2 - y), \quad g(y) = -y$

27. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$

28. $f(y) = \frac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$

29. $f(x) = \frac{10}{x}, \quad x = 0, \quad y = 2, \quad y = 10$

30. $g(x) = \frac{4}{2 - x}, \quad y = 4, \quad x = 0$

Finding the Area of a Region In Exercises 31–36, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region analytically, and (c) use the integration capabilities of the graphing utility to verify your results.

31. $f(x) = x(x^2 - 3x + 3), \quad g(x) = x^2$

32. $y = x^4 - 2x^2, \quad y = 2x^2$

33. $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$

34. $f(x) = x^4 - 9x^2, \quad g(x) = x^3 - 9x$

35. $f(x) = \dfrac{1}{1 + x^2}, \quad g(x) = \dfrac{1}{2}x^2$

36. $f(x) = \dfrac{6x}{x^2 + 1}, \quad y = 0, \quad 0 \le x \le 3$

Finding the Area of a Region In Exercises 37–42, sketch the region bounded by the graphs of the functions and find the area of the region.

37. $f(x) = \cos x, \quad g(x) = 2 - \cos x, \quad 0 \le x \le 2\pi$

38. $f(x) = \sin x, \quad g(x) = \cos 2x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{6}$

39. $f(x) = 2 \sin x, \quad g(x) = \tan x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$

40. $f(x) = \sec \dfrac{\pi x}{4} \tan \dfrac{\pi x}{4}, \quad g(x) = (\sqrt{2} - 4)x + 4, \quad x = 0$

41. $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$

42. $f(x) = 2^x, \quad g(x) = \dfrac{3}{2}x + 1$

Finding the Area of a Region In Exercises 43–46, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

43. $f(x) = 2 \sin x + \sin 2x, \quad y = 0, \quad 0 \le x \le \pi$

44. $f(x) = 2 \sin x + \cos 2x, \quad y = 0, \quad 0 < x \le \pi$

45. $f(x) = \dfrac{1}{x^2}e^{1/x}, \quad y = 0, \quad 1 \le x \le 3$

46. $g(x) = \dfrac{4 \ln x}{x}, \quad y = 0, \quad x = 5$

Finding the Area of a Region In Exercises 47–50, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) explain why the area of the region is difficult to find by hand, and (c) use the integration capabilities of the graphing utility to approximate the area to four decimal places.

47. $y = \sqrt{\dfrac{x^3}{4 - x}}, \quad y = 0, \quad x = 3$

48. $y = \sqrt{x}\, e^x, \quad y = 0, \quad x = 0, \quad x = 1$

49. $y = x^2, \quad y = 4 \cos x$

50. $y = x^2, \quad y = \sqrt{3 + x}$

Integration as an Accumulation Process In Exercises 51–54, find the accumulation function F. Then evaluate F at each value of the independent variable and graphically show the area given by each value of F.

51. $F(x) = \displaystyle\int_0^x \left(\frac{1}{2}t + 1\right) dt$ (a) $F(0)$ (b) $F(2)$ (c) $F(6)$

52. $F(x) = \displaystyle\int_0^x \left(\frac{1}{2}t^2 + 2\right) dt$ (a) $F(0)$ (b) $F(4)$ (c) $F(6)$

53. $F(\alpha) = \displaystyle\int_{-1}^{\alpha} \cos \frac{\pi\theta}{2}\, d\theta$ (a) $F(-1)$ (b) $F(0)$ (c) $F\left(\frac{1}{2}\right)$

54. $F(y) = \displaystyle\int_{-1}^{y} 4e^{x/2}\, dx$ (a) $F(-1)$ (b) $F(0)$ (c) $F(4)$

Finding the Area of a Figure In Exercises 55–58, use integration to find the area of the figure having the given vertices.

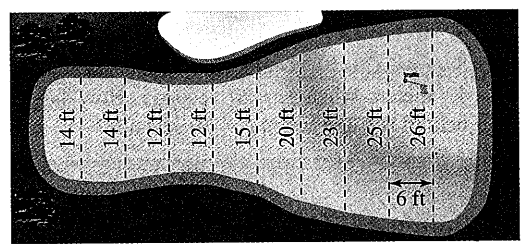
55. $(2, -3), (4, 6), (6, 1)$

56. $(0, 0), (6, 0), (4, 3)$

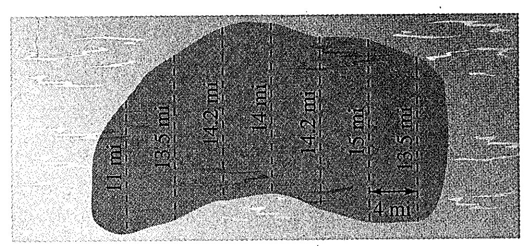
57. $(0, 2), (4, 2), (0, -2), (-4, -2)$

58. $(0, 0), (1, 2), (3, -2), (1, -3)$

59. **Numerical Integration** Estimate the surface area of the golf green using (a) the Trapezoidal Rule and (b) Simpson's Rule.



60. **Numerical Integration** Estimate the surface area of the oil spill using (a) the Trapezoidal Rule and (b) Simpson's Rule.



Using a Tangent Line In Exercises 61–64, set up and evaluate the definite integral that gives the area of the region bounded by the graph of the function and the tangent line to the graph at the given point.

61. $f(x) = x^3, \quad (1, 1)$

62. $y = x^3 - 2x, \quad (-1, 1)$

63. $f(x) = \dfrac{1}{x^2 + 1}, \quad \left(1, \dfrac{1}{2}\right)$

64. $y = \dfrac{2}{1 + 4x^2}, \quad \left(\dfrac{1}{2}, 1\right)$

WRITING ABOUT CONCEPTS

65. Area Between Curves The graphs of $y = 1 - x^2$ and $y = x^4 - 2x^2 + 1$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

66. Using Symmetry The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral $\int_{-1}^{1} (x^3 - x)\, dx$. Explain why this is so. Use symmetry to write a single integral that does represent the area.

67. Interpreting Integrals Two cars with velocities v_1 and v_2 are tested on a straight track (in meters per second). Consider the following.

$$\int_0^5 [v_1(t) - v_2(t)]\, dt = 10 \qquad \int_0^{10} [v_1(t) - v_2(t)]\, dt = 30$$

$$\int_{20}^{30} [v_1(t) - v_2(t)]\, dt = -5$$

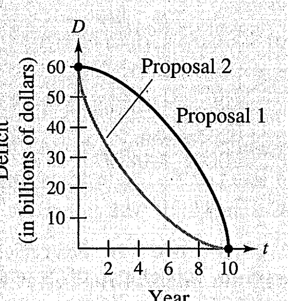
(a) Write a verbal interpretation of each integral.

(b) Is it possible to determine the distance between the two cars when $t = 5$ seconds? Why or why not?

(c) Assume both cars start at the same time and place. Which car is ahead when $t = 10$ seconds? How far ahead is the car?

(d) Suppose Car 1 has velocity v_1 and is ahead of Car 2 by 13 meters when $t = 20$ seconds. How far ahead or behind is Car 1 when $t = 30$ seconds?

68. HOW DO YOU SEE IT? A state legislature is debating two proposals for eliminating the annual budget deficits after 10 years. The rate of decrease of the deficits for each proposal is shown in the figure.



(a) What does the area between the two curves represent?

(b) From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

Dividing a Region **In Exercises 69 and 70, find b such that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.**

69. $y = 9 - x^2, \quad y = 0$ **70.** $y = 9 - |x|, \quad y = 0$

Dividing a Region **In Exercises 71 and 72, find a such that the line $x = a$ divides the region bounded by the graphs of the equations into two regions of equal area.**

71. $y = x, \quad y = 4, \quad x = 0$ **72.** $y^2 = 4 - x, \quad x = 0$

Limits and Integrals **In Exercises 73 and 74, evaluate the limit and sketch the graph of the region whose area is represented by the limit.**

73. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\, \Delta x$, where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$

74. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\, \Delta x$, where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$

Revenue **In Exercises 75 and 76, two models R_1 and R_2 are given for revenue (in billions of dollars) for a large corporation. Both models are estimates of revenues from 2015 through 2020, with $t = 15$ corresponding to 2015. Which model projects the greater revenue? How much more total revenue does that model project over the six-year period?**

75. $R_1 = 7.21 + 0.58t$
$R_2 = 7.21 + 0.45t$

76. $R_1 = 7.21 + 0.26t + 0.02t^2$
$R_2 = 7.21 + 0.1t + 0.01t^2$

77. Lorenz Curve Economists use *Lorenz curves* to illustrate the distribution of income in a country. A Lorenz curve, $y = f(x)$, represents the actual income distribution in the country. In this model, x represents percents of families in the country and y represents percents of total income. The model $y = x$ represents a country in which each family has the same income. The area between these two models, where $0 \le x \le 100$, indicates a country's "income inequality." The table lists percents of income y for selected percents of families x in a country.

x	10	20	30	40	50
y	3.35	6.07	9.17	13.39	19.45

x	60	70	80	90
y	28.03	39.77	55.28	75.12

(a) Use a graphing utility to find a quadratic model for the Lorenz curve.

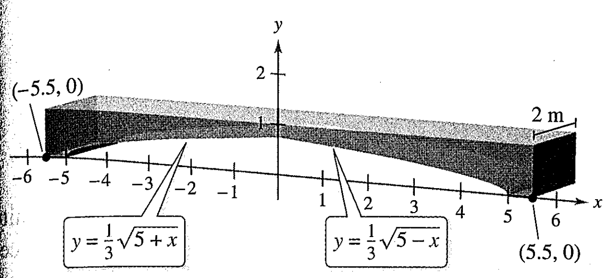
(b) Plot the data and graph the model.

(c) Graph the model $y = x$. How does this model compare with the model in part (a)?

(d) Use the integration capabilities of a graphing utility to approximate the "income inequality."

78. Profit The chief financial officer of a company reports that profits for the past fiscal year were \$15.9 million. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5%. Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.

79. Building Design Concrete sections for a new building have the dimensions (in meters) and shape shown in the figure.

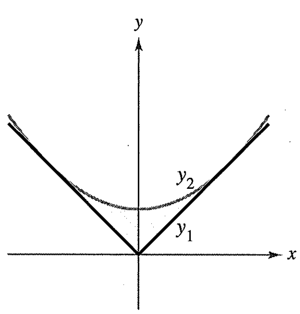


(a) Find the area of the face of the section superimposed on the rectangular coordinate system.

(b) Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.

(c) One cubic meter of concrete weighs 5000 pounds. Find the weight of the section.

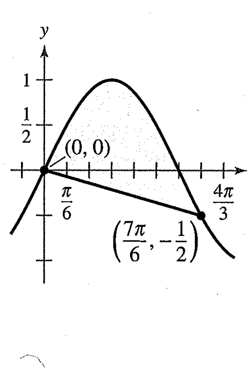
80. Mechanical Design The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).



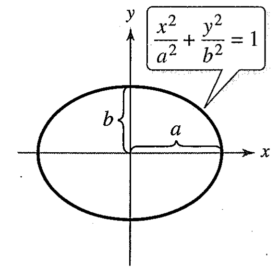
(a) Find k where the parabola is tangent to the graph of y_1.

(b) Find the area of the surface of the machine part.

81. Area Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$, as shown in the figure.



82. Area Let $a > 0$ and $b > 0$. Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab (see figure).



True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

84. If

$$\int_a^b [f(x) - g(x)]\,dx = A$$

then

$$\int_a^b [g(x) - f(x)]\,dx = -A.$$

85. If the graphs of f and g intersect midway between $x = a$ and $x = b$, then

$$\int_a^b [f(x) - g(x)]\,dx = 0.$$

86. The line

$$y = \left(1 - \sqrt[3]{0.5}\right)x$$

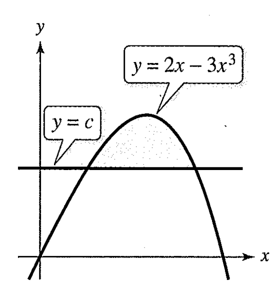
divides the region under the curve

$$f(x) = x(1 - x)$$

on $[0, 1]$ into two regions of equal area.

PUTNAM EXAM CHALLENGE

87. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal.



This problem was composed by the Committee on the Putnam Prize Competition.