(26)

PAGE 542 4. The particular solution of the differential equation $\frac{dy}{dx} = x\sqrt[3]{x^2 - 1}$ with the initial condition, if $x = 3$, then $y = 2$ is

(A) $y = \frac{3}{4}(x^2 - 1)^{4/3} - 10$ $\boxed{\text{(B)}}$ $y = \frac{3}{8}(x^2 - 1)^{4/3} - 4$

(C) $y = \frac{3}{8}(x^2 - 1)^{4/3} + 6$ (D) $y = \frac{3}{4}(x^2 - 1)^{4/3} + 14$

$y = \int x(x^2-1)^{1/3}\,dx$

$u = x^2 - 1$ | $dx = \frac{du}{2x}$

$\frac{du}{dx} = 2x$

$\int \cancel{x} \cdot u^{1/3} \cdot \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{1/3}\,du$

$\frac{1}{2} \cdot \frac{u^{4/3}}{4/3} + C$

$y = \frac{1}{2} \cdot \frac{3}{4}(x^2-1)^{4/3} + C$

$y = \frac{3}{8}(x^2-1)^{4/3} + C$

* plug in $y(3) = 2$ to solve for C

$2 = \frac{3}{8}(3^2-1)^{4/3} + C$

$2 = \frac{3}{8}(8)^{4/3} + C$

$2 = \frac{3}{8}(16) + C$

$2 = 6 + C$

$-4 = C$

$\boxed{y = \frac{3}{8}(x^2-1)^{4/3} + 4}$

7.2 AP Practice Problems

1. Find the solution of the differential equation $\frac{dy}{dx} = \frac{\cos x}{3y^2}$, with the boundary condition $y\left(\frac{\pi}{6}\right) = 1$.

(A) $y^3 = \sin x - \frac{1}{2}$ (B) $y = \sin x + \frac{1}{2}$

$\boxed{\text{(C)}}$ $y^3 = \sin x + \frac{1}{2}$ (D) $y^3 = \sin x + \frac{\sqrt{3}}{2}$

$3y^2\,dy = \cos x\,dx$

$y^2\,dy = \frac{1}{3}\cos x\,dx$

$\int y^2\,dy = \frac{1}{3}\int \cos x\,dx$

$\frac{y^3}{3} + C = \frac{1}{3}\sin x + C$

$\left[\frac{y^3}{3} = \frac{1}{3}\sin x + C\right](3)$

$y^3 = \sin x + C \leftarrow y\left(\frac{\pi}{6}\right) = 1$

$1 = \sin(\pi/6) + C$

$1 = \frac{1}{2} + C$

$\frac{1}{2} = C$

$\boxed{y^3 = \sin x + \frac{1}{2}}$

2. Which of the following is the solution to the differential equation $\frac{dy}{dx} = \frac{x}{y}$, with the initial condition $y(0) = 1$?

$\boxed{\text{(A)}}$ $y = \sqrt{x^2 + 1}$ (B) $y = x^2 + 1$

(C) $y = \pm\sqrt{x^2 + 1}$ (D) $y = -\sqrt{x^2 + 1}$

$y\,dy = x\,dx$

$(2)\left[\frac{y^2}{2} = \frac{x^2}{2} + C\right]$

$y^2 = x^2 + C \leftarrow$ plug in $(0,1)$

$1^2 = 0^2 + C$

$1 = C$

$y^2 = x^2 + 1$

$y = \pm\sqrt{x^2+1}$

y-value is positive

$\boxed{y = \sqrt{x^2+1}}$

3. Suppose $\frac{dy}{dx} = e^y \cos x$, and $y = 0$ when $x = \pi$.

Then evaluate y when $x = \frac{\pi}{6}$.

(A) $\ln\frac{1}{2}$ (B) $\boxed{\ln 2}$ (C) $\ln\left(1-\frac{\sqrt{3}}{2}\right)^{-1}$ (D) $\frac{1}{2}$

$\frac{dy}{dx} = \frac{e^y \cos x}{1}$

$dy = e^y \cos x \, dx$

$\frac{dy}{e^y} = \cos x \, dx$

$\int e^{-y} dy = \int \cos x \, dx$

$-e^{-y} = \sin x + C$

$-\frac{1}{e^y} = \sin x + c$ ← $y(\pi)=0$

$\frac{-1}{e^0} = \sin(\pi) + C$

$-1 = \sin\pi + C$

$-1 = C$

$-\frac{1}{e^y} = \sin x - 1$

$-\frac{1}{e^y} = \sin(\pi/6) - 1$

$\frac{-1}{e^y} = \frac{1}{2} - 1$

$\frac{-1}{e^y} = \frac{-1}{2}$

$e^y = 2$

$\ln e^y = \ln 2$

$\boxed{y = \ln 2}$

4. Solve $\frac{dy}{dx} = x^3 y$. Then y equals

(A) $\frac{4}{Cx^4}$ (B) $\frac{x^4}{4} + C$ (C) Ce^{3x^2} (D) $\boxed{Ce^{x^4/4}}$

$\frac{dy}{dx} = \frac{x^3 y}{1}$

$dy = x^3 y \, dx$

$\frac{dy}{y} = x^3 dx$

$\int \frac{1}{y} dy = \int x^3 dx$

$\ln|y| = \frac{x^4}{4} + C$

$e^{\ln|y|} = e^{x^4/4 + c}$

$|y| = e^{x^4/4} \cdot e^c$

$y = e^{x^4/4} \cdot C$

$\boxed{y = Ce^{x^4/4}}$

5. If $\frac{dy}{dx} = 5y^2$ and $y = 1$ when $x = 3$, then find y when $x = 0$.

(A) $-\frac{1}{6}$ (B) $-\frac{1}{16}$ (C) $\boxed{\frac{1}{16}}$ (D) $\frac{1}{6}$

$\frac{dy}{dx} = \frac{5y^2}{1}$

$dy = 5y^2 dx$

$\frac{dy}{y^2} = 5\,dx$

$\int y^{-2} dy = \int 5\,dx$

$\frac{y^{-1}}{-1} = 5x + c$

$\frac{-1}{y} = 5x + c$ ← $y(3)=1$

$-\frac{1}{1} = 5(3) + c$

$-1 = 15 + c$

$-16 = c$

$\boxed{-\frac{1}{y} = 5x - 16}$

plug in $x=0$

$-\frac{1}{y} = 5(0) - 16$

$-\frac{1}{y} = -16$

$\boxed{y = \frac{1}{16}}$

6. If $\frac{dy}{dx} = \frac{y}{1+x^2}$ and $y = 1$ if $x = -1$, then y equals

(A) $\frac{\pi}{4} e^{\tan^{-1} x}$ (B) $e^{\tan^{-1} x} + \frac{\pi}{4}$

(C) $e^{\tan^{-1} x} + e^{\pi/4}$ (D) $\boxed{e^{\tan^{-1} x + \pi/4}}$

$(1+x^2) dy = y\,dx$

$\frac{dy}{y} = \frac{dx}{1+x^2}$

$\int \frac{1}{y} dy = \int \frac{1}{1+x^2} dx$

$\ln|y| = \frac{1}{1}\arctan\left(\frac{x}{1}\right) + C$

$e^{\ln|y|} = e^{\arctan x + c}$

$|y| = e^{\arctan x} \cdot e^c$

$|y| = Ce^{\arctan x}$ ← plug in $y(-1)=1$

$1 = Ce^{\arctan(-1)}$

$1 = Ce^{-\pi/4}$

$\frac{1}{e^{-\pi/4}} = C$

$e^{\pi/4} = C$

$y = e^{\pi/4} \cdot e^{\arctan x}$

$\boxed{y = e^{\arctan x + \pi/4}}$

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7. A population of insects increases according to the uninhibited growth equation $\frac{dP}{dt} = kP$, where k is a constant and t is time in days. If the population doubles every 12 days, then k equals

(A) $\frac{\ln 2}{12}$ (B) $\frac{(\ln 2)^2}{\ln 12}$

(C) $(\ln 2)\ln 12$ (D) $\log_2 12$

$P = Ce^{kt}$

(t, P)
$(0, C)$
$(12, 2C)$

$2C = Ce^{k(12)}$
$2 = e^{12k}$
$\ln 2 = \ln e^{12k}$
$\ln 2 = 12k \ln e$
$\ln 2 = 12k$

$\boxed{\frac{\ln 2}{12} = k}$

8. Suppose $\frac{dA}{dt} = k(100 - A)$, where $k > 0$ is a constant and $A < 100$. If $A = A_0$ when $t = 0$, then

(A) $A = A_0 e^{kt}$ (B) $A = (100 - A_0)e^{-kt}$

(C) $A = 100 - (100 - A_0)e^{-kt}$ (D) $A = (100 - A_0)e^{-100kt}$

$dA = k(100 - A)dt$

$\int \frac{dA}{100 - A} = \int k\,dt$

$-\ln|100 - A| = kt + C$
$\ln|100 - A| = -kt + C$
$e^{\ln|100-A|} = e^{-kt} \cdot e^{c}$

$|100 - A| = Ce^{-kt}$
$100 - A = Ce^{-kt}$
$100 - Ce^{-kt} = A$

plug in $(0, A_0)$

$100 - Ce^{-k(0)} = A_0$
$100 - A_0 = C$

$\boxed{A = 100 - (100 - A_0)e^{-kt}}$

9. A colony of bacteria is growing at a rate $\frac{dB}{dt} = 6e^{3t/4}$ grams per hour. If initially there are 8 grams of bacteria in the colony, how many grams will be present in 12 hours?

(A) 12 g (B) 72 g (C) $6e^9$g (D) $8e^9$g

$dB = 6e^{3t/4}dt$

$B = \int 6e^{3t/4}dt$

$u = \frac{3t}{4}$
$\frac{du}{dt} = 3/4$
$dt = \frac{4}{3}du$

$\int 6e^u \cdot \frac{4}{3}du$
$\int 8e^u du$

$B = 8e^{3t/4} + C$
$8 = 8e^0 + C$
$0 = C$
$B = 8e^{3t/4}$
$B(12) = 8e^{\frac{3(12)}{4}} = \boxed{8e^9 \text{ grams}}$

(t, B)
$(0, 8)$
$(12, \_)$

10. An apple pie is baked to a temperature of 400°F then placed on a rack to cool in a room with a constant temperature of 70°F. After 20 min the temperature of the pie is 300°F. To the nearest degree, what is the temperature of the pie after 60 min?

T_s = surrounding temp
T_0 = initial temp

(A) 86°F (B) 100°F (C) 182°F (D) 190°F

$T - T_s = (T_0 - T_s)e^{-kt}$
$T - 70 = (400 - 70)e^{-kt}$
$T = (330)e^{-kt} + 70$

(t, T)
$(20, 300)$
$(60, \_)$

$300 = 330e^{-k(20)} + 70$
$230 = 330e^{-20k}$
$0.697 = e^{-20k}$
$\ln 0.697 = \ln e^{-20k}$

$-20k = \ln 0.697$
$k = 0.018$
$T = 330e^{-0.018t} + 70$
$T = 330e^{-0.018(60)} + 70$

$\boxed{T \approx 182°}$