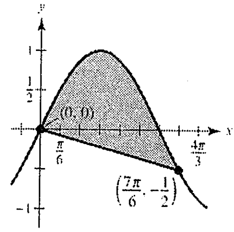
81. Line: $y = \dfrac{-3}{7\pi}x$

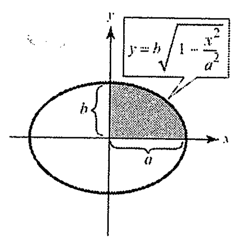
$$A = \int_0^{7\pi/6}\left[\sin x + \frac{3x}{7\pi}\right]dx$$
$$= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6}$$
$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$
$$\approx 2.7823$$



82. $A = 4\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\,dx = \dfrac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

$\int_0^a \sqrt{a^2 - x^2}\,dx$ is the area of $\frac{1}{4}$ of a circle $= \dfrac{\pi a^2}{4}$.

So, $A = \dfrac{4b}{a}\left(\dfrac{\pi a^2}{4}\right) = \pi ab.$



83. True. The region has been shifted C units upward (if $C > 0$), or C units downward (if $C < 0$).

84. True. This is a property of integrals.

85. False. Let $f(x) = x$ and $g(x) = 2x - x^2$, f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$$\int_a^b\left[f(x) - g(x)\right]dx = \int_0^2\left[x - \left(2x - x^2\right)\right]dx = \tfrac{2}{3} \neq 0.$$

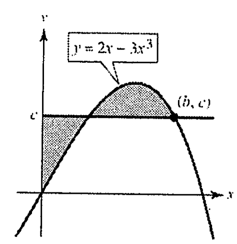
86. True. The area under $f(x)$ between 0 and 1 is $\frac{1}{6}$. The curves intersect at $x = \frac{1}{2}^{1/3}$, and the area between $y = \left(1 - \frac{1}{2}^{1/3}\right)x$ and f on the interval $\left[0, \frac{1}{2}^{1/3}\right]$ is $\frac{1}{12}$.

87. You want to find c such that:

$$\int_0^b\left[\left(2x - 3x^3\right) - c\right]dx = 0$$
$$\left[x^2 - \tfrac{3}{4}x^4 - cx\right]_0^b = 0$$
$$b^2 - \tfrac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because (b, c) is on the graph.

$$b^2 - \tfrac{3}{4}b^4 - \left(2b - 3b^3\right)b = 0$$
$$4 - 3b^2 - 8 + 12b^2 = 0$$
$$9b^2 = 4$$
$$b = \tfrac{2}{3}$$
$$c = \tfrac{4}{9}$$



Section 7.2 Volume: The Disk Method

1. $V = \pi\int_0^1(-x + 1)^2\,dx = \pi\int_0^1\left(x^2 - 2x + 1\right)dx = \pi\left[\dfrac{x^3}{3} - x^2 + x\right]_0^1 = \dfrac{\pi}{3}$

2. $V = \pi\int_0^2\left(4 - x^2\right)^2dx = \pi\int_0^2\left(x^4 - 8x^2 + 16\right)dx = \pi\left[\dfrac{x^5}{5} - \dfrac{8x^3}{3} + 16x\right]_0^2 = \dfrac{256\pi}{15}$

3. $V = \pi\int_1^4\left(\sqrt{x}\right)^2dx = \pi\int_1^4 x\,dx = \pi\left[\dfrac{x^2}{2}\right]_1^4 = \dfrac{15\pi}{2}$

4. $V = \pi\int_0^3\left(\sqrt{9 - x^2}\right)^2dx = \pi\int_0^3\left(9 - x^2\right)dx = \pi\left[9x - \dfrac{x^3}{3}\right]_0^3 = 18\pi$

5. $V = \pi \int_0^1 \left[\left(x^2\right)^2 - \left(x^5\right)^2 \right] dx$

$= \pi \int_0^1 \left(x^4 - x^{10}\right) dx$

$= \pi \left[\frac{x^5}{5} - \frac{x^{11}}{11} \right]_0^1$

$= \pi \left(\frac{1}{5} - \frac{1}{11} \right) = \frac{6\pi}{55}$

6. $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2 \right] dx$

$= 2\pi \int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12 \right] dx$

$= 2\pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}}$

$= 2\pi \left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right]$

$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$

7. $y = x^2 \Rightarrow x = \sqrt{y}$

$V = \pi \int_0^4 \left(\sqrt{y}\right)^2 dy = \pi \int_0^4 y\, dy$

$= \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$

8. $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$

$V = \pi \int_0^4 \left(\sqrt{16 - y^2}\right)^2 dy = \pi \int_0^4 \left(16 - y^2\right) dy$

$= \pi \left[16y - \frac{y^3}{3} \right]_0^4 = \frac{128\pi}{3}$

9. $y = x^{2/3} \Rightarrow x = y^{3/2}$

$V = \pi \int_0^1 \left(y^{3/2}\right)^2 dy = \pi \int_0^1 y^3\, dy = \pi \left[\frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$

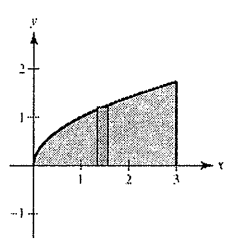
10. $V = \pi \int_1^4 \left(-y^2 + 4y\right)^2 dy = \pi \int_1^4 \left(y^4 - 8y^3 + 16y^2\right) dy$

$= \pi \left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$

11. $y = \sqrt{x},\ y = 0,\ x = 3$

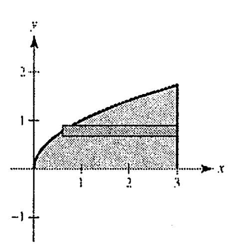
(a) $R(x) = \sqrt{x},\ r(x) = 0$

$V = \pi \int_0^3 \left(\sqrt{x}\right)^2 dx = \pi \int_0^3 x\, dx = \pi \left[\frac{x^2}{2} \right]_0^3 = \frac{9\pi}{2}$



(b) $R(y) = 3,\ r(y) = y^2$

$V = \pi \int_0^{\sqrt{3}} \left[3^2 - \left(y^2\right)^2 \right] dy = \pi \int_0^{\sqrt{3}} \left(9 - y^4\right) dy = \pi \left[9y - \frac{y^5}{5} \right]_0^{\sqrt{3}} = \pi \left[9\sqrt{3} - \frac{9}{5}\sqrt{3} \right] = \frac{36\sqrt{3}\pi}{5}$

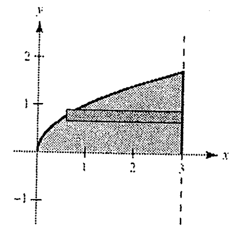


(c) $R(y) = 3 - y^2, r(y) = 0$

$$V = \pi \int_0^{\sqrt{3}} \left(3 - y^2\right)^2 dy = \pi \int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) dy$$

$$= \pi\left[9y - 2y^3 + \frac{y^5}{5}\right]_0^{\sqrt{3}} = \pi\left[9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}\right]$$

$$= \frac{24\sqrt{3}\pi}{5}$$

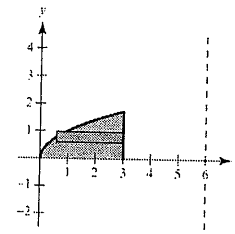


(d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2, r(y) = 3$

$$V = \pi \int_0^{\sqrt{3}} \left[\left(6 - y^2\right)^2 - 3^2\right] dy = \pi \int_0^{\sqrt{3}} \left(y^4 - 12y^2 + 27\right) dy$$

$$= \pi\left[\frac{y^5}{5} - 4y^3 + 27y\right]_0^{\sqrt{3}} = \pi\left[\frac{9\sqrt{3}}{5} - 12\sqrt{3} + 27\sqrt{3}\right]$$

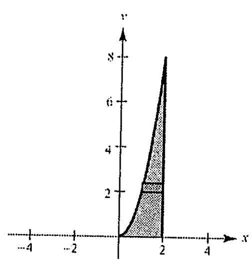
$$= \frac{84\sqrt{3}\pi}{5}$$



12. $y = 2x^2, y = 0, x = 2$

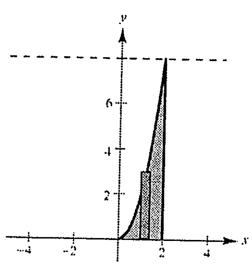
(a) $R(y) = 2, r(y) = \sqrt{y/2}$

$$V = \pi \int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$



(b) $R(x) = 2x^2, r(x) = 0$

$$V = \pi \int_0^2 4x^4\, dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$



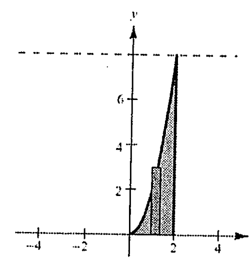
(c) $R(x) = 8, r(x) = 8 - 2x^2$

$$V = \pi \int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right] dx$$

$$= \pi \int_0^2 \left(32x^2 - 4x^4\right) dx = 4\pi \int_0^2 \left(8x^2 - x^4\right) dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$

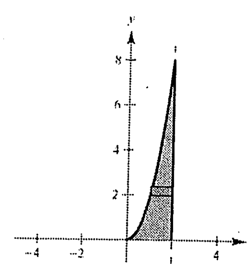


(d) $R(y) = 2 - \sqrt{y/2}, r(y) = 0$

$$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi \int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

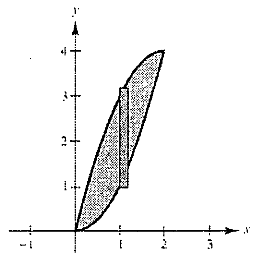
$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$



13. $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

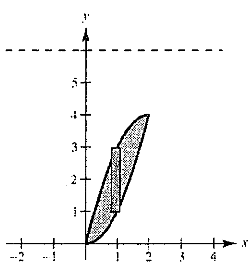
(a) $R(x) = 4x - x^2$, $r(x) = x^2$

$$V = \pi\int_0^2\left[\left(4x - x^2\right)^2 - x^4\right]dx$$
$$= \pi\int_0^2\left(16x^2 - 8x^3\right)dx$$
$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$



(b) $R(x) = 6 - x^2$, $r(x) = 6 - \left(4x - x^2\right)$

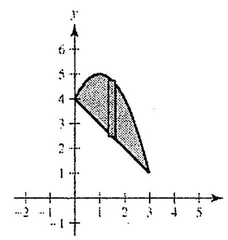
$$V = \pi\int_0^2\left[\left(6 - x^2\right)^2 - \left(6 - 4x + x^2\right)^2\right]dx$$
$$= 8\pi\int_0^2\left(x^3 - 5x^2 + 6x\right)dx$$
$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



14. $y = 4 + 2x - x^2$, $y = 4 - x$ intersect at $(0, 4)$ and $(3, 1)$.

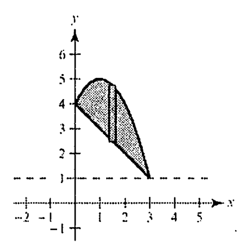
(a) $R(x) = 4 + 2x - x^2$, $r(x) = 4 - x$

$$V = \pi\int_0^3\left[\left(4 + 2x - x^2\right)^2 - (4 - x)^2\right]dx$$
$$= \pi\int_0^3\left(x^4 - 4x^3 - 5x^2 + 24x\right)dx$$
$$= \pi\left[\frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2\right]_0^3 = \frac{153\pi}{5}$$



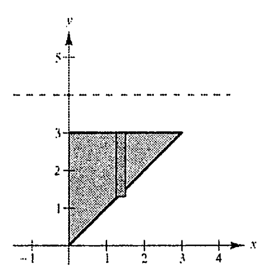
(b) $R(x) = \left(4 + 2x - x^2\right) - 1$, $r(x) = (4 - x) - 1$

$$V = \pi\int_0^3\left[\left(3 + 2x - x^2\right)^2 - (3 - x)^2\right]dx$$
$$= \pi\int_0^3\left(x^4 - 4x^3 - 3x^2 + 18x\right)dx$$
$$= \pi\left[\frac{x^5}{5} - x^4 - x^3 + 9x^2\right]_0^3 = \frac{108\pi}{5}$$



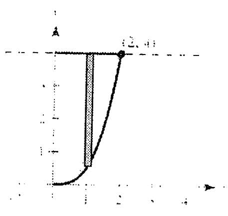
15. $R(x) = 4 - x$, $r(x) = 1$

$$V = \pi\int_0^3\left[(4 - x)^2 - (1)^2\right]dx$$
$$= \pi\int_0^3\left(x^2 - 8x + 15\right)dx$$
$$= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3$$
$$= 18\pi$$



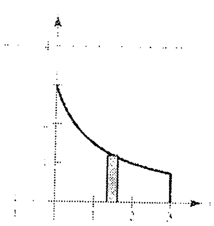
16. $R(x) = 4 - \dfrac{x^3}{2},\ r(x) = 0$

$$V = \pi\int_0^1\left(4 - \frac{x^3}{2}\right)^2 dx$$
$$= \pi\int_0^2\left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$
$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2$$
$$= \pi\left(32 - 16 + \frac{128}{28}\right)$$
$$= \frac{144}{7}\pi$$



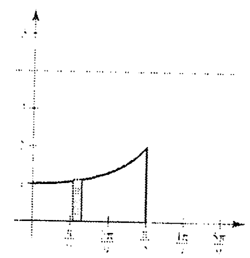
17. $R(x) = 4,\ r(x) = 4 - \dfrac{3}{1 + x}$

$$V = \pi\int_0^3\left[4^2 - \left(4 - \frac{3}{1 + x}\right)^2\right] dx$$
$$= \pi\int_0^3\left[\frac{24}{1 + x} - \frac{9}{(1 + x)^2}\right] dx$$
$$= \pi\left[24 \ln|1 + x| + \frac{9}{1 + x}\right]_0^3$$
$$= \pi\left[\left(24 \ln 4 + \frac{9}{4}\right) - 9\right]$$
$$= \left(48 \ln 2 - \frac{27}{4}\right)\pi \approx 83.318$$



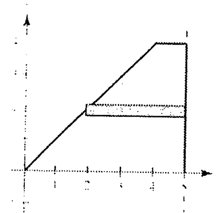
18. $R(x) = 4,\ r(x) = 4 - \sec x$

$$V = \pi\int_0^{\pi/3}\left[(4)^2 - (4 - \sec x)^2\right] dx$$
$$= \pi\int_0^{\pi/3}\left(8 \sec x - \sec^2 x\right) dx$$
$$= \pi\left[8 \ln|\sec x + \tan x| - \tan x\right]_0^{\pi/3}$$
$$= \pi\left[\left(8 \ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - \left(8 \ln|1 + 0| - 0\right)\right]$$
$$= \pi\left[8 \ln\left(2 + \sqrt{3}\right) - \sqrt{3}\right] \approx 27.66$$



19. $R(y) = 5 - y,\ r(y) = 0$

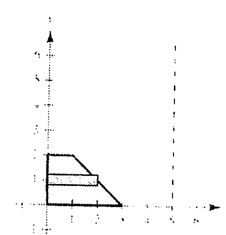
$$V = \pi\int_0^4 (5 - y)^2\, dy$$
$$= \pi\int_0^4 \left(25 - 10y + y^2\right) dy$$
$$= \pi\left[25y - 5y^2 + \frac{y^3}{3}\right]_0^4$$
$$= \pi\left[100 - 80 + \frac{64}{3}\right]$$
$$= \frac{124\pi}{3}$$



20. $y = 3 - x,\ x = 3 - y$

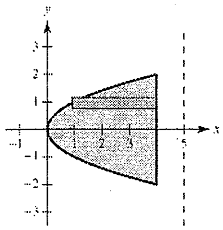
$R(y) = 5,\ r(y) = 5 - (3 - y) = 2 + y$

$$V = \pi\int_0^2\left[5^2 - (2 + y)^2\right] dy$$
$$= \pi\int_0^2\left(-y^2 - 4y + 21\right) dy$$
$$= \pi\left[\frac{-y^3}{3} - 2y^2 + 21y\right]_0^2 = \frac{94\pi}{3}$$



21. $R(y) = 5 - y^2, r(y) = 1$

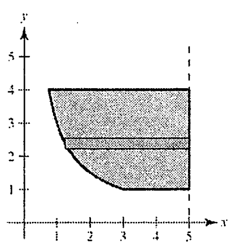
$$V = \pi \int_{-2}^{2} \left[\left(5 - y^2\right)^2 - 1\right] dy$$
$$= 2\pi \int_0^2 \left[y^4 - 10y^2 + 24\right] dy$$
$$= 2\pi\left[\frac{y^5}{5} - \frac{10y^3}{3} + 24y\right]_0^2$$
$$= 2\pi\left[\frac{32}{5} - \frac{80}{3} + 48\right] = \frac{832\pi}{15}$$



22. $xy = 3, x = \frac{3}{y}$

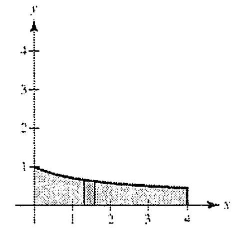
$R(y) = 5 - \frac{3}{y}, r(y) = 0$

$$V = \pi \int_1^4 \left(5 - \frac{3}{y}\right)^2 dy$$
$$= \pi \int_1^4 \left(25 + \frac{9}{y^2} - \frac{30}{y}\right) dy$$
$$= \pi\left[25y - \frac{9}{y} - 30 \ln y\right]_1^4$$
$$= \pi\left[\left(100 - \frac{9}{4} - 30 \ln 4\right) - (25 - 9)\right]$$
$$= \pi\left[\frac{327}{4} - 30 \ln 4\right] \approx 126.17$$



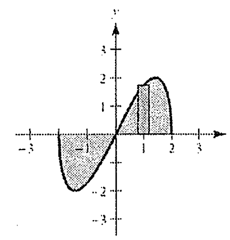
23. $R(x) = \frac{1}{\sqrt{x + 1}}, r(x) = 0$

$$V = \pi \int_0^4 \left(\frac{1}{\sqrt{x + 1}}\right)^2 dx$$
$$= \pi \int_0^4 \frac{1}{x + 1} dx = \pi\left[\ln|x + 1|\right]_0^4 = \pi \ln 5$$



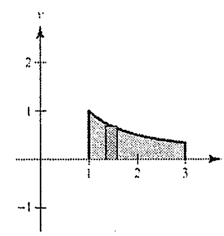
24. $R(x) = x\sqrt{4 - x^2}, r(x) = 0$

$$V = 2\pi \int_0^2 \left(x\sqrt{4 - x^2}\right)^2 dx$$
$$= 2\pi \int_0^2 \left(4x^2 - x^4\right) dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= 2\pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{128\pi}{15}$$



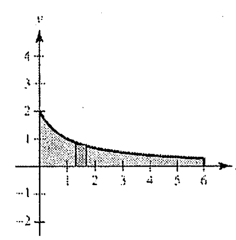
25. $R(x) = \frac{1}{x}, r(x) = 0$

$$V = \pi \int_1^3 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^3$$
$$= \pi\left[-\frac{1}{3} + 1\right] = \frac{2}{3}\pi$$



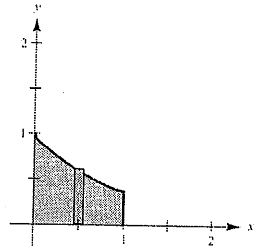
26. $R(x) = \frac{2}{x + 1}, r(x) = 0$

$$V = \pi \int_0^6 \left(\frac{2}{x + 1}\right)^2 dx$$
$$= 4\pi \int_0^6 (x + 1)^{-2} dx$$
$$= 4\pi\left[\frac{-1}{x + 1}\right]_0^6$$
$$= 4\pi\left[-\frac{1}{7} + 1\right] = \frac{24\pi}{7}$$



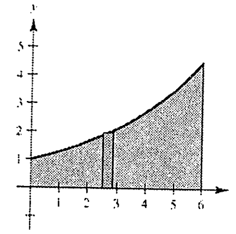
27. $R(x) = e^{-x}, r(x) = 0$

$$V = \pi \int_0^1 \left(e^{-x}\right)^2 dx$$
$$= \pi \int_0^1 e^{-2x}\, dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}\left(1 - e^{-2}\right) \approx 1.358$$



28. $R(x) = e^{x/4}, r(x) = 0$

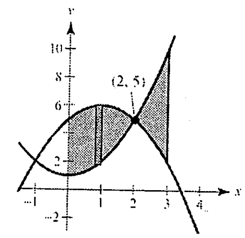
$$V = \pi \int_0^6 \left(e^{x/4}\right)^2 dx$$
$$= \pi \int_0^6 e^{x/2}\, dx$$
$$= \pi\left[2e^{x/2}\right]_0^6$$
$$= \pi\left(2e^3 - 2\right) \approx 119.92$$



29.
$$x^2 + 1 = -x^2 + 2x + 5$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$

The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi \int_0^2 \left[\left(5 + 2x - x^2\right)^2 - \left(x^2 + 1\right)^2\right] dx + \pi \int_2^3 \left[\left(x^2 + 1\right)^2 - \left(5 + 2x - x^2\right)^2\right] dx$$
$$= \pi \int_0^2 \left(-4x^3 - 8x^2 + 20x + 24\right) dx + \pi \int_2^3 \left(4x^3 + 8x^2 - 20x - 24\right) dx$$
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$
$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

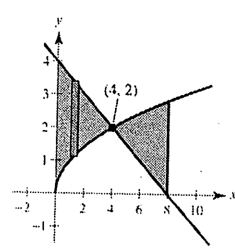


30. $\sqrt{x} = -\frac{1}{2}x + 4$

$$x = \frac{1}{4}x^2 - 4x + 16$$
$$0 = x^2 - 20x + 64$$
$$0 = (x - 4)(x - 16)$$

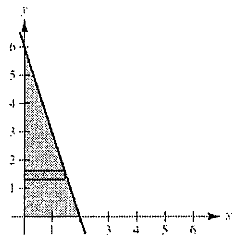
The curves intersect at $(4, 2)$. (Note $x = 16$ is an extraneous root.)

$$V = \pi \int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi \int_4^8 \left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$$
$$= \pi \int_0^4 \left(\frac{x^2}{4} - 5x + 16\right) dx + \pi \int_4^8 \left(-\frac{x^2}{4} + 5x - 16\right) dx$$
$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$
$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$



31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

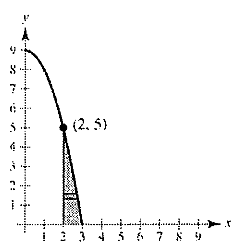
$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^6 \left[36 - 12y + y^2\right] dy$$
$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$
$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$
$$= 8\pi = \frac{1}{3}\pi r^2 h, \text{ Volume of cone}$$



32. $y = 9 - x^2, y = 0, x = 2, x = 3$

$x = \sqrt{9 - y}$

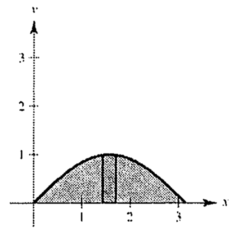
$$V = \pi\int_0^5 \left[\left(\sqrt{9 - y}\right)^2 - 2\right]^2 dy$$
$$= \pi\int_0^5 (5 - y)\, dy$$
$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5 = \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$



33. $V = \pi\int_0^\pi (\sin x)^2\, dx$

$$= \pi\int_0^\pi \frac{1 - \cos 2x}{2}\, dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

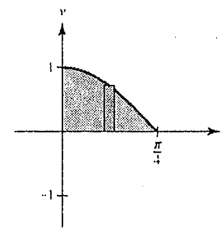
Numerical approximation: 4.9348



34. $V = \pi\int_0^{\pi/4} \cos^2 2x\, dx$

$$= \pi\int_0^{\pi/4} \frac{1 + \cos 4x}{2}\, dx$$
$$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\pi/4}$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

Numerical approximation: 1.2337



35. $V = \pi\int_1^2 \left(e^{x-1}\right)^2 dx$

$$= \pi\int_1^2 e^{2x-2}\, dx$$
$$= \frac{\pi}{2}e^{2x-2}\Big]_1^2$$
$$= \frac{\pi}{2}\left(e^2 - 1\right)$$

Numerical approximation: 10.0359

36. $V = \pi\int_{-1}^2 \left[e^{x/2} + e^{-x/2}\right]^2 dx$

$$= \pi\int_{-1}^2 \left[e^x + e^{-x} + 2\right] dx$$
$$= \pi\left[e^x - e^{-x} + 2x\right]_{-1}^2$$
$$= \pi\left[\left(e^2 - e^{-2} + 4\right) - \left(e^{-1} - e - 2\right)\right]$$
$$= \pi\left(e^2 + e + 6 - e^{-2} - e^{-1}\right)$$

Numerical approximation: 49.0218

37. $V = \pi\int_0^2 \left[e^{-x^2}\right]^2 dx \approx 1.9686$

38. $V = \pi\int_1^3 [\ln x]^2\, dx \approx 3.2332$

39. $V = \pi\int_0^5 \left[2\arctan(0.2x)\right]^2 dx$

≈ 15.4115

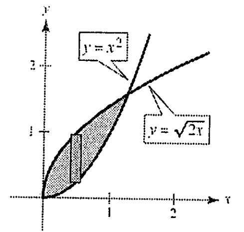
40. $x^2 = \sqrt{2x}$

$x^4 = 2x$

$x^3 = 2$

$x = 2^{1/3} \approx 1.2599$

$$V = \pi \int_0^{2^{1/3}} \left[\left(\sqrt{2x}\right)^2 - \left(x^2\right)^2\right] dx \approx 2.9922$$



41. $V = \pi \int_0^1 y^2\, dy = \pi \dfrac{y^3}{3}\Big]_0^1 = \dfrac{\pi}{3}$

42. $V = \pi \int_0^1 \left[1^2 - (1 - y)^2\right] dy$

$= \pi \int_0^1 \left[2y - y^2\right] dy$

$= \pi \left[y^2 - \dfrac{y^3}{3}\right]_0^1$

$= \pi\left(1 - \dfrac{1}{3}\right) = \dfrac{2}{3}\pi$

43. $V = \pi \int_0^1 \left(x^2 - x^4\right) dx$

$= \pi \left[\dfrac{x^3}{3} - \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\dfrac{1}{3} - \dfrac{1}{5}\right)$

$= \dfrac{2\pi}{15}$

44. $V = \pi \int_0^1 \left[\left(1 - x^2\right)^2 - (1 - x)^2\right] dx$

$= \pi \int_0^1 \left[1 - 2x^2 + x^4 - 1 + 2x - x^2\right] dx$

$= \pi \int_0^1 \left[2x - 3x^2 + x^4\right] dx$

$= \pi \left[x^2 - x^3 + \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\dfrac{1}{5}\right) = \dfrac{\pi}{5}$

45. $V = \pi \int_0^1 (1 - y)\, dy$

$= \pi \left[y - \dfrac{y^2}{2}\right]_0^1 = \pi\left(1 - \dfrac{1}{2}\right) = \dfrac{\pi}{2}$

46. $V = \pi \int_0^1 \left(1 - \sqrt{y}\right)^2 dy$

$= \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$

$= \pi \left[y - \dfrac{4}{3}y^{3/2} + \dfrac{y^2}{2}\right]_0^1$

$= \pi\left(1 - \dfrac{4}{3} + \dfrac{1}{2}\right)$

$= \dfrac{\pi}{6}$

47. $V = \pi \int_0^1 \left(y - y^2\right) dy$

$= \pi \left[\dfrac{y^2}{2} - \dfrac{y^3}{3}\right]_0^1 = \pi\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{\pi}{6}$

48. $V = \pi \int_0^1 \left[(1 - y)^2 - \left(1 - \sqrt{y}\right)^2\right] dy$

$= \pi \int_0^1 \left[1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right] dy$

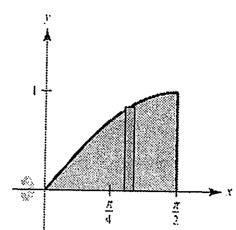
$= \pi \int_0^1 \left[2\sqrt{y} - 3y + y^2\right] dy$

$= \pi \left[\dfrac{4}{3}y^{3/2} - \dfrac{3y^2}{2} + \dfrac{y^3}{3}\right]_0^1$

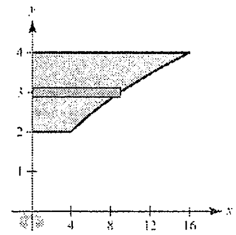
$= \pi\left(\dfrac{4}{3} - \dfrac{3}{2} + \dfrac{1}{3}\right)$

$= \dfrac{\pi}{6}$

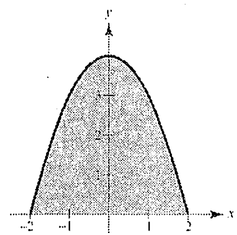
49. $\pi \int_0^{\pi/2} \sin^2 x\, dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.

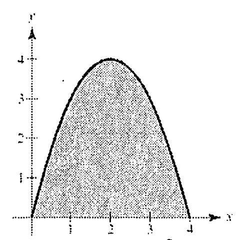


50. $\pi \int_2^4 y^4\, dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2, x = 0, y = 2, y = 4$ about the y-axis.



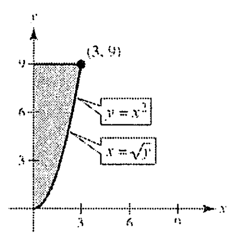
51.





The volumes are the same because the solid has been translated horizontally. $\left(4x - x^2 = 4 - (x - 2)^2\right)$

52.



(a) Around x-axis:

$$V = \pi \int_0^3 \left[9^2 - \left(x^2\right)^2\right] dx = \frac{972}{5}\pi = 194.4\pi$$

(b) Around y-axis:

$$V = \pi \int_0^9 \left(\sqrt{y}\right)^2 dy = \frac{81}{2}\pi = 40.5\pi$$

(c) Around $x = 3$:

$$V = \pi\left(3^2\right)9 - \int_0^9 \pi\left(\sqrt{y} - 3\right)^2 dy = 81\pi - \frac{27}{2}\pi$$

$$= \frac{135\pi}{2} \approx 67.5\pi$$

So, $b < c < a$.

53. (a) True. Answers will vary.

(b) False. Answers will vary.

54. (a) Matches (ii) because the axis of rotation is vertical, and this is the washer method.

(b) Matches (iv) because the axis of rotation is horizontal, and this is the washer method.

(c) Matches (i) because the axis of rotation is horizontal.

(d) Matches (iii) because the axis of rotation is vertical.

55. $V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx = \pi \int_0^4 x\, dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi \int_0^c x\, dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

So, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

56. Set $\pi \int_0^c x\, dx = \frac{8\pi}{3}$ (one third of the volume).

Then $\frac{\pi c^2}{2} = \frac{8\pi}{3}, c^2 = \frac{16}{3}, c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$.

To find the other value, set $\pi \int_0^d x\, dx = \frac{16\pi}{3}$

(two thirds of the volume).

Then $\frac{\pi d^2}{2} = \frac{16\pi}{3}, d^2 = \frac{32}{3}, d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}$.

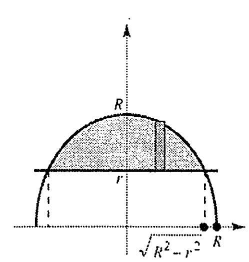
The x-values that divide the solid into three parts of equal volume are $x = \left(4\sqrt{3}\right)/3$ and $x = \left(4\sqrt{6}\right)/3$.

57. $V = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - r^2 - x^2\right) dx$$

$$= 2\pi\left[\left(R^2 - r^2\right)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \frac{\left(R^2 - r^2\right)^{3/2}}{3}\right] = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$$



58. Let $R = 6$ in the previous Exercise.

$$\frac{4}{3}\pi\left(36 - r^2\right)^{3/2} = \frac{1}{2}\left(\frac{4}{3}\right)\pi(6)^3$$

$$\left(36 - r^2\right)^{3/2} = 108$$

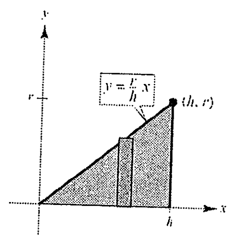
$$36 - r^2 = (108)^{2/3}$$

$$r^2 = 36 - 108^{2/3}$$

$$r = \sqrt{36 - 108^{2/3}} \approx 3.65$$

59. $R(x) = \dfrac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx = \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h = \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



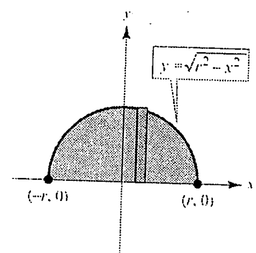
60. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$V = \pi\int_{-r}^{r}\left(r^2 - x^2\right)dx$$

$$= 2\pi\int_0^r\left(r^2 - x^2\right)dx$$

$$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$$

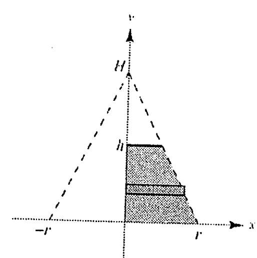


61. $x = r - \dfrac{r}{H}y = r\left(1 - \dfrac{y}{H}\right),\ R(y) = r\left(1 - \dfrac{y}{H}\right),\ r(y) = 0$

$$V = \pi\int_0^h\left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2\int_0^h\left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right)dy$$

$$= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h$$

$$= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right) = \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$$



62. $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

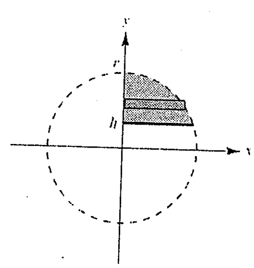
$$V = \pi\int_h^r\left(\sqrt{r^2 - y^2}\right)^2 dy$$

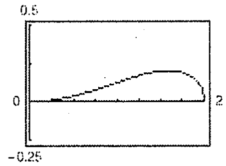
$$= \pi\int_h^r\left(r^2 - y^2\right)dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right]$$

$$= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right) = \frac{\pi}{3}\left(2r^3 - 3r^2h + h^3\right)$$



63. 

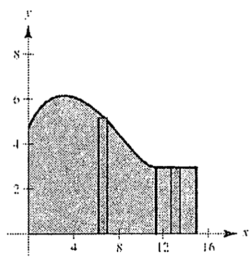
$$V = \pi\int_0^2\left(\frac{1}{8}x^2\sqrt{2-x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2-x)\,dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}\text{ m}^3$$

64. $y = \begin{cases}\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5\\ 2.95, & 11.5 < x \le 15\end{cases}$

$$V = \pi\int_0^{11.5}\left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\,dx$$

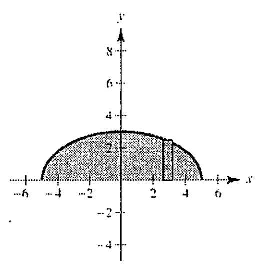
$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2x\right]_{11.5}^{15}$$

$\approx$ 1031.9016 cubic centimeters



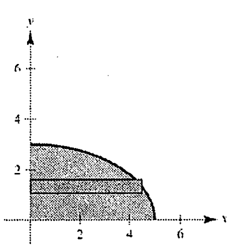
65. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2}$, $r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^{5}\left(25 - x^2\right)dx = \frac{18\pi}{25}\int_0^5\left(25 - x^2\right)dx = \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = 60\pi$$



(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2}$, $r(y) = 0$, $x \ge 0$

$$V = \frac{25\pi}{9}\int_0^3\left(9 - y^2\right)dy = \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3 = 50\pi$$

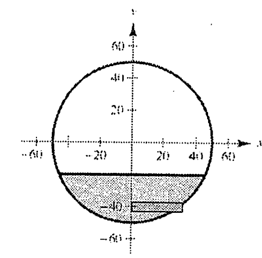


66. Total volume: $V = \dfrac{4\pi(50)^3}{3} = \dfrac{500{,}000\pi}{3}\text{ ft}^3$

Volume of water in the tank:

$$\pi\int_{-50}^{y_0}\left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0}\left(2500 - y^2\right)dy = \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0} = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:



$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

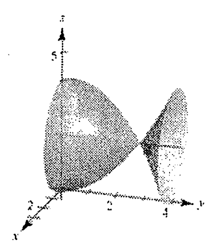
When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

67. (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$

$$x^2 = 16 - 4b = 4(4 - b)$$

$$x = 2\sqrt{4 - b}$$



$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^{4} \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$

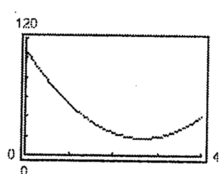
$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$

$$= \pi\int_0^4\left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right]dx$$

$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$

$$= \pi\left(\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$

(b) Graph of $V(b) = \pi\left(4b^2 - \dfrac{64}{3}b + \dfrac{512}{15}\right)$



Minimum volume is 17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left(8b - \dfrac{64}{3}\right) = 0 \Rightarrow b = \dfrac{64/3}{8} = \dfrac{8}{3} = 2\dfrac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \dfrac{8}{3}$ is a relative minimum.

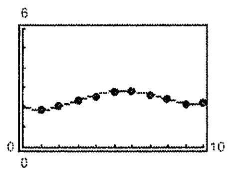
68. (a) $V = \int_0^{10} \pi[f(x)]^2\,dx$

Simpson's Rule: $b - a = 10 - 0 = 10,\ n = 10$

$$V \approx \frac{\pi}{3}\Big[(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2\Big]$$

$$\approx \frac{\pi}{3}(178.405) \approx 186.83 \text{ cm}^3$$

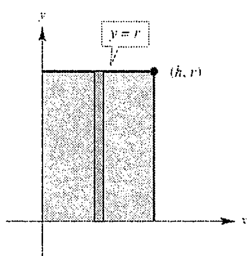
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c) $V \approx \int_0^{10} \pi f(x)^2\,dx \approx 186.35 \text{ cm}^3$

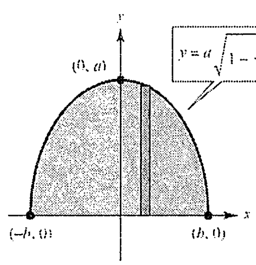
69. (a) $\pi \int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



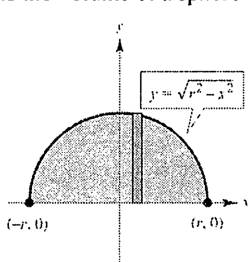
(b) $\pi \int_{-b}^{b} \left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



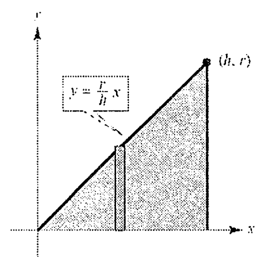
(c) $\pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



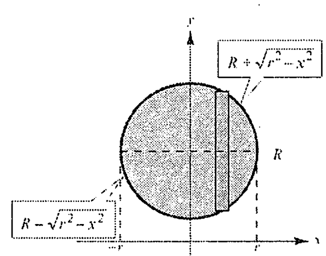
(d) $\pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.



(e) $\pi \int_{-r}^{r} \left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.

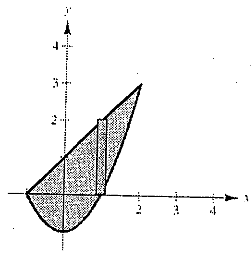


70. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Because $A_1(x) = A_2(x)$, you have

$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2.$$

So, the volumes are the same.

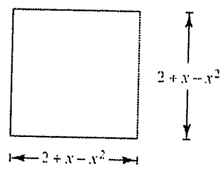
71.



Base of cross section $= (x + 1) - (x^2 - 1) = 2 + x - x^2$

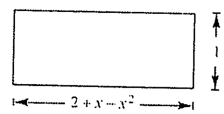
(a) $A(x) = b^2 = (2 + x - x^2)^2 = 4 + 4x - 3x^2 - 2x^3 + x^4$

$$V = \int_{-1}^{2} (4 + 4x - 3x^2 - 2x^3 + x^4)\,dx = \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

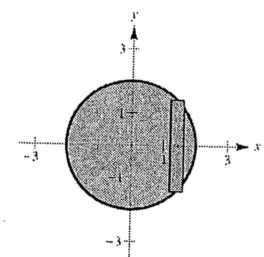


(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



72.



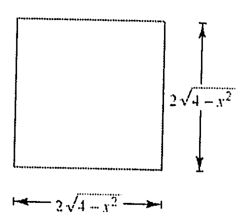
Base of cross section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4(4 - x^2)\,dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{128}{3}$$

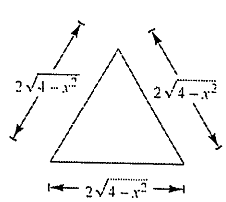


(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right) = \sqrt{3}(4 - x^2)$

$$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\,dx$$

$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

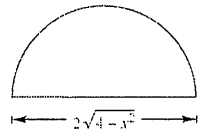
$$= \frac{32\sqrt{3}}{3}$$



(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}\left(4 - x^2\right)$

$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4 - x^2\right)dx$$

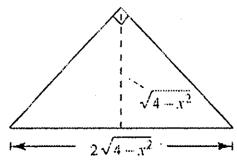
$$= \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$$



(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$$V = \int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$



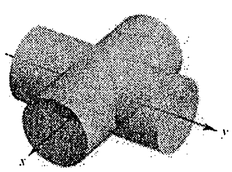
73. The cross sections are squares. By symmetry, you can set up an integral for an eighth of the volume and multiply by 8.

$$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$$

$$V = 8\int_0^r\left(r^2 - y^2\right)dy$$

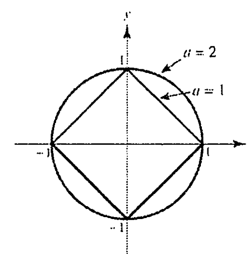
$$= 8\left[r^2 y - \tfrac{1}{3}y^3\right]_0^r$$

$$= \tfrac{16}{3}r^3$$



74. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.



(b) $|y| = \left(1 - |x|^a\right)^{1/a}$

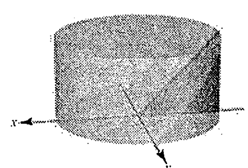
$$A = 2\int_{-1}^{1}\left(1 - |x|^a\right)^{1/a}dx = 4\int_0^1\left(1 - x^a\right)^{1/a}dx$$

To approximate the volume of the solid, from n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.

75. (a) Because the cross sections are isosceles right triangles:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}\left(r^2 - y^2\right)$$

$$V = \frac{1}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \int_0^r\left(r^2 - y^2\right)dy = \left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$$



(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \frac{\tan\theta}{2}\left(r^2 - y^2\right)$

$$V = \frac{\tan\theta}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy = \tan\theta\int_0^r\left(r^2 - y^2\right)dy = \tan\theta\left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3\tan\theta$$

As $\theta \to 90°$, $V \to \infty$.