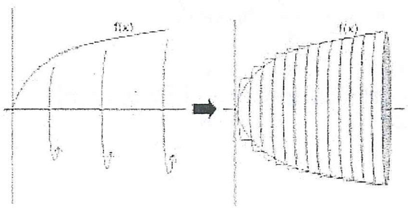
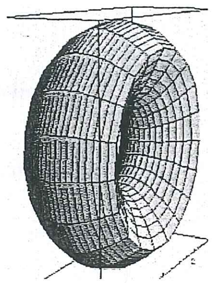
7.2c Volumes with Known Cross Section

Cross – Section is the shape that results if we cut the object in half and look at the resulting shape sideways:

Disc Method



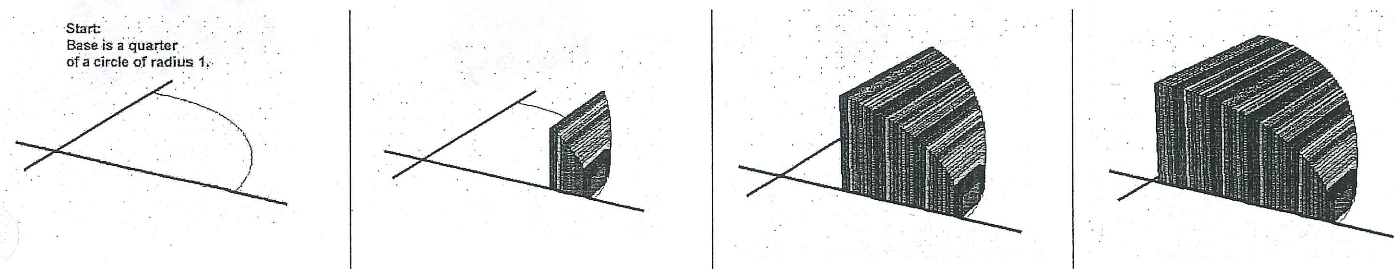
Washer Method



$$V = \pi \int [Area \text{ of cross section}]\, dx$$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR).

Now, we still have a shaded region, but now it will act as the base. We will now build the cross-section area on top of this base.



Top/Bottom Vertical radius (rectangles):	Right/Left Horizontal radius (rectangles)
$V = \int_{x_1}^{x_2} [Area \text{ of cross section}]dx$	$V = \int_{y_1}^{y_2} [Area \text{ of cross section}]dy$
*Note: All values in integral are in terms of x	*Note: All values in integral are in terms of y

Steps for Finding Volume with known cross sections

1. Graph equations
2. Shade the enclosed region (this is the base of the 3D figure)
3. Determine Top/Bottom (dx) or Right/Left (dy) method
   1. If cross section is perpendicular to x-axis (or parallel to y-axis),then Top/Bottom method (dx)
   2. If cross section is perpendicular to y-axis (or parallel to x-axis), then Right/Left method (dy)
4. Draw rectangle between 2 curves
   1. Many times the distance between the 2 curves will act as the base or radius or hypotenuse depending on your cross section.
5. Find Area for cross section using the formulas:
6. (If necessary) Set equations equal to each other to find intersecting end points
7. Plug values into definite integral to find Volume

Area for cross section using below formulas:

1. Squares : Area = (base) * (base)
2. Isosceles Right Triangles: Area $= \frac{1}{2}(base) * (base)$

   *since base = height
3. Isosceles Right Triangle(Alternate) Area

   $\frac{1}{2}\left(\frac{hypotenuse}{\sqrt{2}}\right)^2$ or $\frac{1}{2}\left(\frac{hypotenuse^2}{2}\right) = \frac{1}{4}(hypotenuse)^2$
4. Equilateral Triangles: Area $= \frac{\sqrt{3}}{4}(base)^2$
5. Semicircles: Area $= \frac{\pi}{2}[Radius]^2$
6. Semicircles : Area $= \frac{\pi}{8}[Diameter]^2$

Top/Bottom Vertical radius (rectangles):	Right/Left Horizontal radius (rectangles)
$V = \int_{x_1}^{x_2} [Area \text{ of cross section}]dx$	$V = \int_{y_1}^{y_2} [Area \text{ of cross section}]dy$
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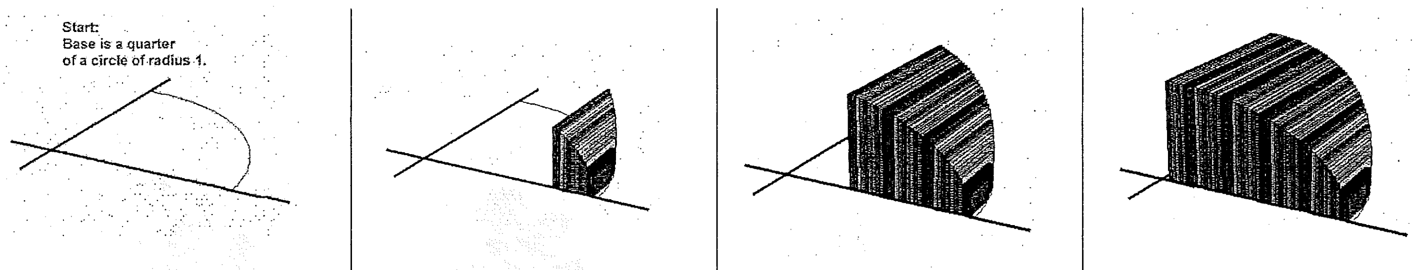
5. Semicircles: Area $= \frac{1}{2}\pi[Radius]^2 = \frac{\pi}{8}[Diameter]^2$

Example 1: Find the volume of the solid if the base is bounded by curve y = $\sqrt{1-x^2}$, y = 0, x = 0 (in the first quadrant) and the cross sections are squares parallel to the y-axis.



Example 2: Find the volume of the solid if the base is bounded by the curve y = x^2 and the line y = 4 and the cross sections are isosceles right triangles whose hypotenuse lie on the base and are parallel to the x-axis.

Example 3: Find the volume of the solid if the base is bounded by the curve y = x^2 and the line y = 4 and the cross sections are semicircles whose base are perpendicular to the y-axis.

7.2c Volumes with Known Cross Section

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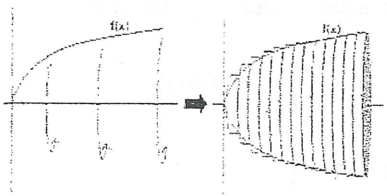
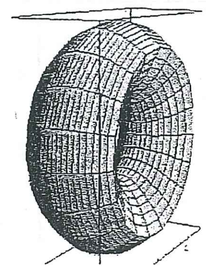
Disc Method



Washer Method



$$V = \pi \int [\mathit{Area} \text{ of cross section}]\, dx$$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR).

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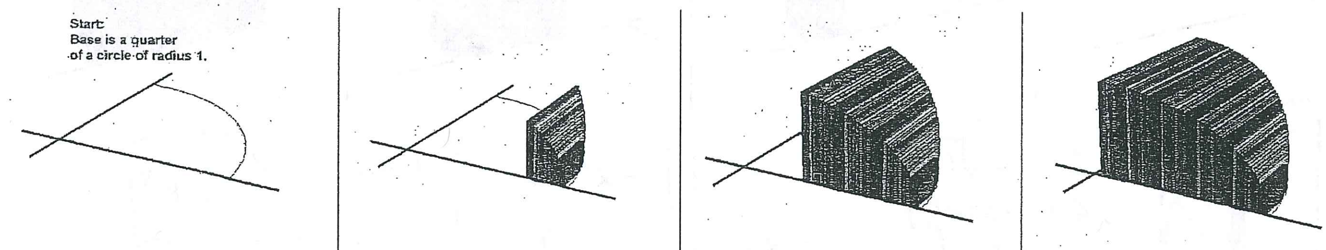

Top/Bottom Vertical radius (rectangles):	Right/Left Horizontal radius (rectangles)
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Area for cross section using below formulas:

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   *since base = height
3. Isosceles Right Triangle(Alternate) Area =

   $\frac{1}{2}\left(\frac{hypotenuse}{\sqrt{2}}\right)^2$ or $\frac{1}{2}\left(\frac{hypotenuse^2}{2}\right)$
4. Equilateral Triangles: Area $= \frac{\sqrt{3}}{4}(base)^2$
5. Semicircles: Area $= \frac{1}{2}\pi[Radius]^2$

Top/Bottom Vertical radius (rectangles):	Right/Left Horizontal radius (rectangles)
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Area for cross section using below formulas:

1. Squares : Area = (base) * (base)
2. Isosceles Right Triangles: Area $= \frac{1}{2}(base)*(base)$ *since base = height
3. Isosceles Right Triangle(Alternate) Area $= \frac{1}{2}\left(\frac{hypotenuse^2}{2}\right) = \frac{1}{4}(hypotenuse)^2$
4. Equilateral Triangles: Area $= \frac{\sqrt{3}}{4}(base)^2$
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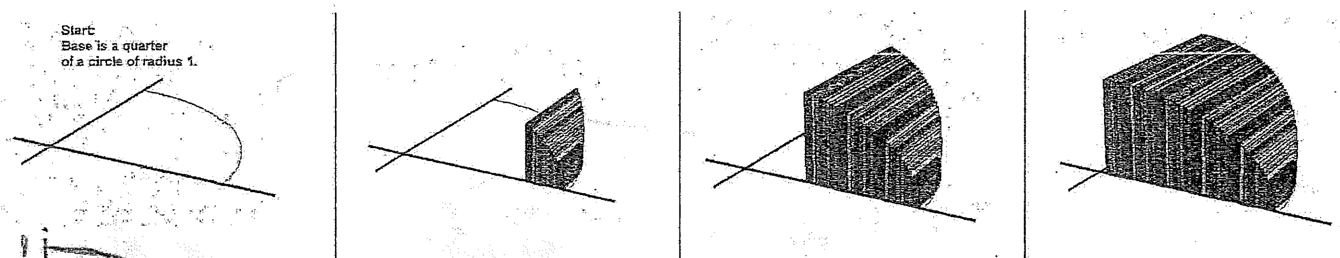
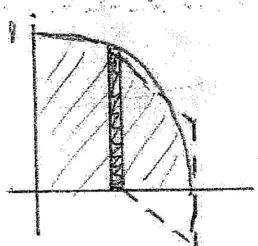
$A = \frac{\pi}{8}[diameter]^2$

Example 1: Find the volume of the solid if the base is bounded by curve y = $\sqrt{1-x^2}$, y = 0, x = 0 (in the first quadrant) and the cross sections are squares parallel to the y-axis.





$base = \sqrt{1-x^2} - 0$

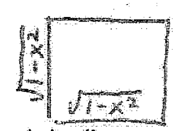
$Area = (base)^2$

$V = \int_0^1 \left(\sqrt{1-x^2}\right)^2 dx$

$\int 1 - x^2\, dx$

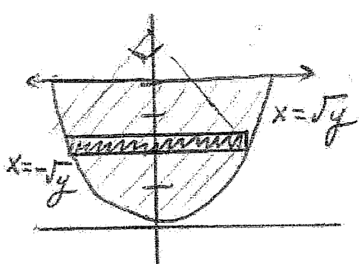
$x - \frac{x^3}{3}\Big]_0^1 =$

$1 - \frac{1}{3} = \boxed{\frac{2}{3} \text{ units}^3}$



$* \sqrt{1-x^2} = 0 \quad x = 1$

Example 2: Find the volume of the solid if the base is bounded by the curve y = x^2 and the line y = 4 and the cross sections are isosceles right triangles whose hypotenuse lie on the base and are parallel to the x-axis.



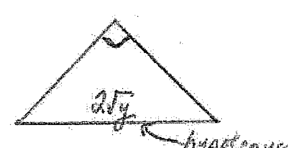
$hypotenuse = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$

$V = \int_0^4 \frac{1}{4}(2\sqrt{y})^2 dy$

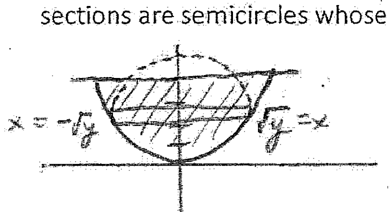
$= \int_0^4 \frac{1}{4}(4y)\, dy$

$Area = \frac{1}{4}(hypotenuse)^2$

$\int_0^4 y\, dy = \frac{y^2}{2}\Big]_0^4 = \frac{16}{2} - 0 = \boxed{8 \text{ units}^3}$



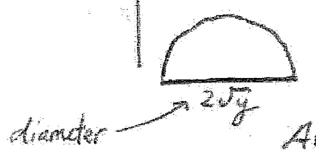
Example 3: Find the volume of the solid if the base is bounded by the curve y = x^2 and the line y = 4 and the cross sections are semicircles whose base are perpendicular to the y-axis.



$diameter = \sqrt{y} - (-\sqrt{y})$

$= 2\sqrt{y}$

$V = \frac{\pi}{8}\int_0^4 (2\sqrt{y})^2 dy$



$Area = \frac{\pi}{8}(diameter)^2$

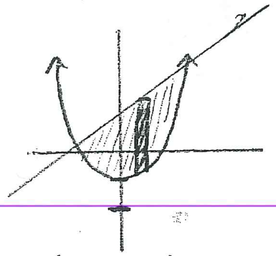
$\frac{\pi}{8}\int_0^4 4y\, dy = \frac{4y^2}{2}\Big]_0^4 = 32 - 0$

$\left(\frac{\pi}{8}\right)32 = \boxed{4\pi \text{ units}^3}$

7.2c Homework p. 465-466 #61, 62, 63

* Volume of Known Cross Sections

61) $y = x+1$, $y = x^2-1$, perpendicular to x-axis



a) Squares

base $= x+1-(x^2-1)$

$= x+1-x^2+1$

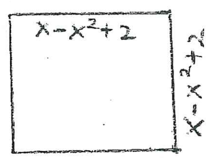
$= x-x^2+2$

* find bounds

$x^2-1 = x+1$

$x^2-x-2=0$

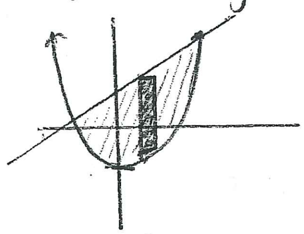
$(x-2)(x+1)=0$ $\boxed{x=2,-1}$



$$V=\int_{-1}^{2}(\text{base})^2\,dx = \int_{-1}^{2}\left[x-x^2+2\right]^2 dx$$

$$= \boxed{\frac{81}{10} \text{ units}^3}$$

b) Rectangle of height 1



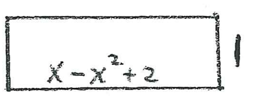
Area $= (\text{base})(1)$

base $= x+1-(x^2-1)$

$= x-x^2+2$

* bounds:

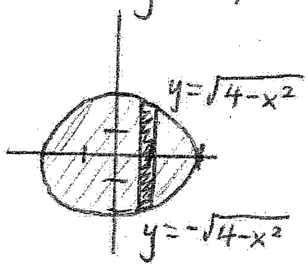
$x=-1, 2$



$$V=\int_{-1}^{2}(x^2-x+2)(1)\,dx = \boxed{\frac{9}{2} \text{ units}^3}$$

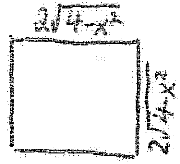
7.2c HW continued

62) $x^2+y^2=4$ perpendicular to x-axis



base $=\sqrt{4-x^2}-(-\sqrt{4-x^2})=2\sqrt{4-x^2}$

a) Squares

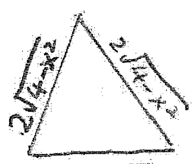


$$V=\int_{-2}^{2}\left(2\sqrt{4-x^2}\right)^2 dx$$

$$\boxed{V=\frac{128}{3}\text{ units}^3}$$

b) Equilateral triangles

base $=2\sqrt{4-x^2}$



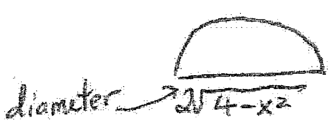
$$\text{Area}=\frac{\sqrt{3}}{4}(\text{base})^2$$

$$V=\frac{\sqrt{3}}{4}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2 dx$$

$$V=\frac{32\sqrt{3}}{3}\approx\boxed{18.475}$$

c) Semicircles

base $=2\sqrt{4-x^2}$



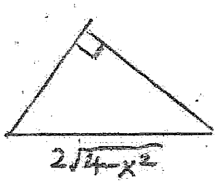
$$A=\frac{\pi}{8}(\text{diameter})^2$$

$$V=\frac{\pi}{8}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2 dx$$

$$\boxed{V=\frac{16}{3}\pi}$$

d) Isosceles right triangles

base $=2\sqrt{4-x^2}$



hypotenuse $=2\sqrt{4-x^2}$

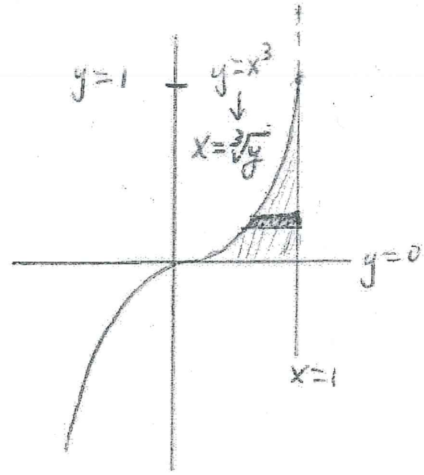
$$A=\frac{1}{4}(\text{hypotenuse})^2$$
$$=\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$$

$$V=\int_{-2}^{2}\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$$

$$\boxed{V=\frac{32}{3}\text{ units}^3}$$

7.2c HW (continued)

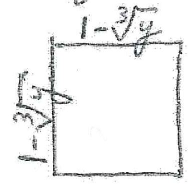
63) graph bounded by $y=x^3$, $y=0$, $x=1$
(perpendicular to the y-axis)



base $= 1-\sqrt[3]{y}$

*find bounds
$\sqrt[3]{y}=1$, $y=1$

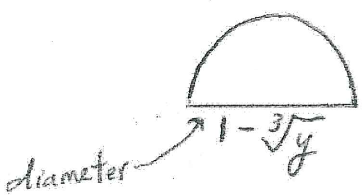
a) squares



$$V=\int_0^1 (1-\sqrt[3]{y})^2\,dy$$

$$\boxed{V=\frac{1}{10}}$$

b) semicircle

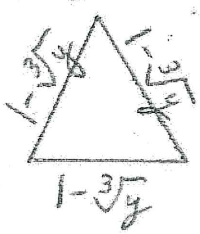


$$A=\frac{\pi}{8}(\text{diameter})^2$$

$$V=\frac{\pi}{8}\int_0^1 [1-\sqrt[3]{y}]^2$$

$$V=\frac{\pi}{8}\left(\frac{1}{10}\right)=\boxed{\frac{\pi}{80}\text{ units}^3}$$

c) Equilateral triangles



$$A=\frac{\sqrt{3}}{4}(\text{base})^2$$

$$V=\frac{\sqrt{3}}{4}\int_0^1 [1-\sqrt[3]{y}]^2\,dy$$

$$V=\frac{\sqrt{3}}{4}\left(\frac{1}{10}\right)$$

$$=\boxed{\frac{\sqrt{3}}{40}\text{ units}^3}$$