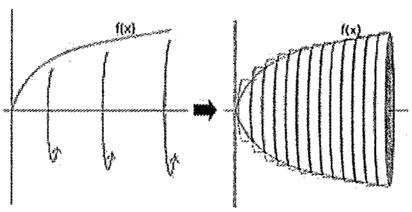
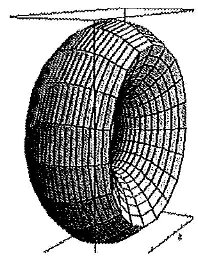
Key

AP Calculus Ch. 7.2c Volumes with Known Cross Section

Cross-section is the shape that results if we cut the object in half and look at the resulting shape sideways:

Disc Method | Washer Method

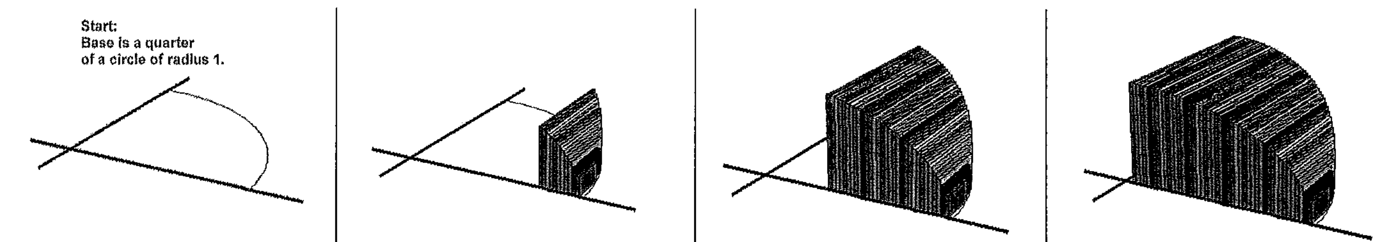




$V = \pi \int [Area \text{ of cross section}]\, dx$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR), resulting in circular Area, either $Area = \pi[R(x)]^2 \text{ or } Area = \pi[R(x)]^2 - \pi[r(x)]^2$

Now, we still have a shaded region, but now it will act as the base of the 3D object. We will now build the cross-section area on top of this base. (no longer rotating around an axis)

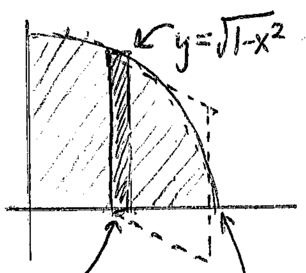


Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [\boldsymbol{Area} \text{ of cross section}]dx$	$V = \int_{y_1}^{y_2} [\boldsymbol{Area} \text{ of cross section}]dy$
*Note: All values in integral are in terms of x (in the form of " y = ____")	*Note: All values in integral are in terms of y (in the forms of " x = ______")

Areas formulas for for Cross- sections:

1. Square: $A = (base)^2$	2. Isosceles Right Triangle (**leg** on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

Example 1: Find the volume of the solid if the base is bounded by curve y = $\sqrt{1-x^2}$, y = 0, x = 0 (in the first quadrant) and the cross sections are squares parallel to the y-axis.



$y = \sqrt{1-x^2}$

base (Top-Bottom) $= \sqrt{1-x^2} - 0 = \sqrt{1-x^2}$

$Area = (base)^2 = (\sqrt{1-x^2})^2$

$y=0$

intersection:

$\sqrt{1-x^2} = 0 \quad | \quad x^2 = 1$

$1 - x^2 = 1 \quad | \quad x = 1$

$V = \int_{x_1}^{x_2} (Area)\, dx$

$V = \int_0^1 [\sqrt{1-x^2}]^2 dx$

$\boxed{V = \frac{2}{3} \text{ units}^3}$

Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [\boldsymbol{Area} \text{ of cross section}]dx$ *Note: All values in integral are in terms of x (equations in the form of " y = ____ ")	$V = \int_{y_1}^{y_2} [\boldsymbol{Area} \text{ of cross section}]dy$ *Note: All values in integral are in terms of y (equations in the form of " x = ____ ")

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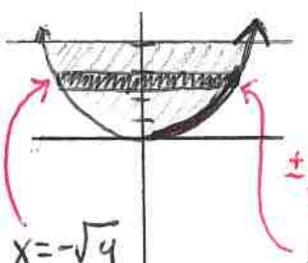


base (right-left): $\sqrt{y} - (-\sqrt{y})$

$y = x^2$

$\pm\sqrt{y} = \sqrt{x^2}$

base $= 2\sqrt{y}$

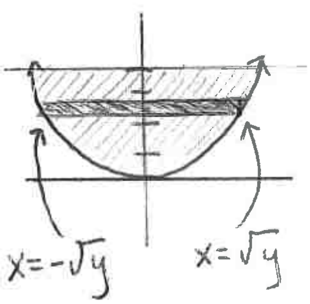
$V = \int_{y_1}^{y_2} [Area]dy$

Area $= \frac{1}{4}\left[2\sqrt{y}\right]^2 = \frac{1}{4}\cdot 4y = y$

$V = \int_0^4 y\,dy \rightarrow \frac{y^2}{2}\Big]_0^4 = \frac{4^2}{2} - 0 = 8$

$\boxed{V = 8 \text{ units}^3}$

Example 3: Find the volume of the solid if the base is bounded by the curve y = x^2 and the line y = 4 and the cross sections are semicircles whose base are perpendicular to the y-axis.



base (right-left) : $\sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$

$A = \frac{\pi}{8}(base)^2$

$A = \frac{\pi}{8}(2\sqrt{y})^2 = \frac{\pi}{8}\cdot 4y$

$A = \frac{\pi}{2}y$

$V = \int_{y_1}^{y_2} Area\; dy$

$V = \frac{\pi}{2}\int_0^4 y\,dy \rightarrow \frac{y^2}{2}\Big]_0^4 = \frac{16}{2} = 8$

$\boxed{V = \frac{\pi}{2}(8) = 4\pi \text{ units}^3}$

Example 4: Let the region R be the area enclosed by the function $f(x) = \ln x$ and $g(x) = \frac{1}{2}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in the region R, find the volume of the solid.

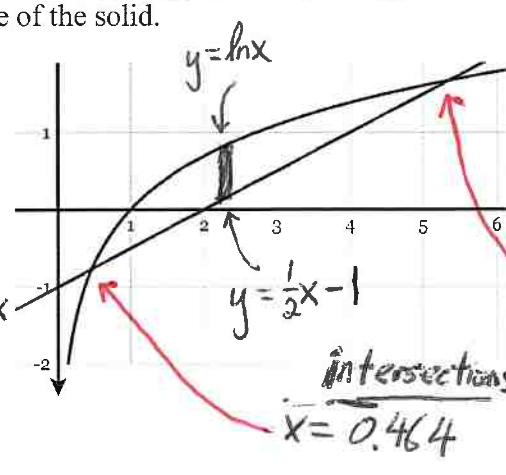
base (top-bottom) : $\ln x - \left(\frac{1}{2}x - 1\right)$

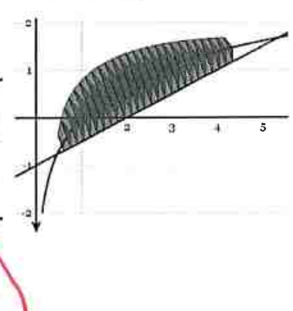
base $= \ln x - \frac{1}{2}x + 1$

Area $= \frac{1}{2}(base)^2 \rightarrow \frac{1}{2}\left(\ln x - \frac{1}{2}x + 1\right)^2$

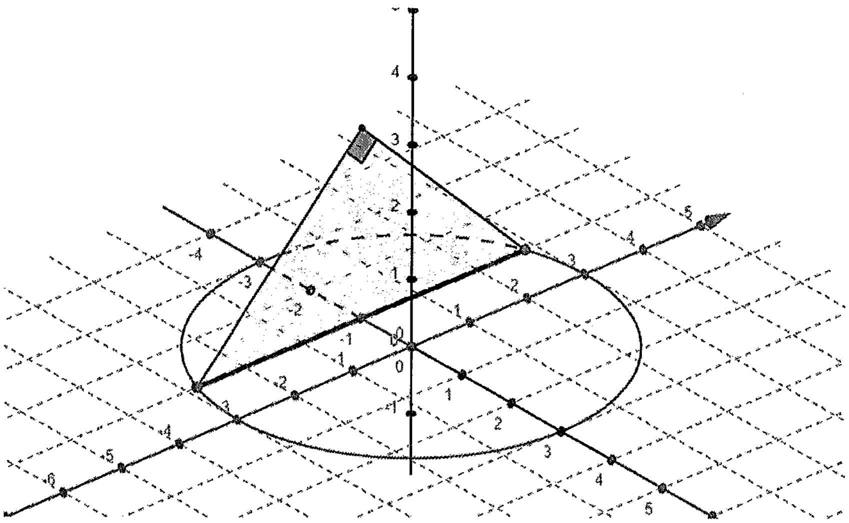
$V = \int_{x_1}^{x_2} [Area]dx \rightarrow V = \int_{0.464}^{5.357} \frac{1}{2}\left(\ln x - \frac{1}{2}x + 1\right)^2 dx$

$\boxed{V = 0.613}$

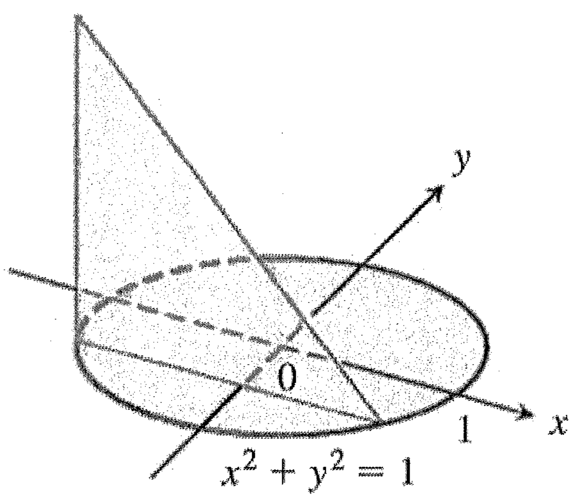




Right Triangle with hypotenuse on the base



Right Triangle with leg on the base



<u>AP Calculus Ch. 7.2c Volumes with Known Cross Section</u>

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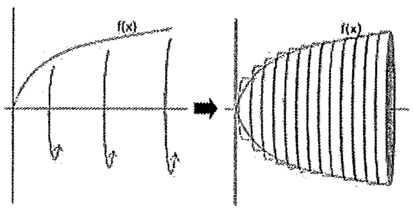
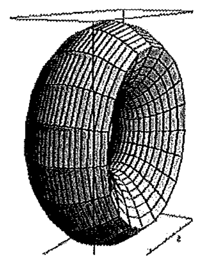


<u>Washer Method</u>



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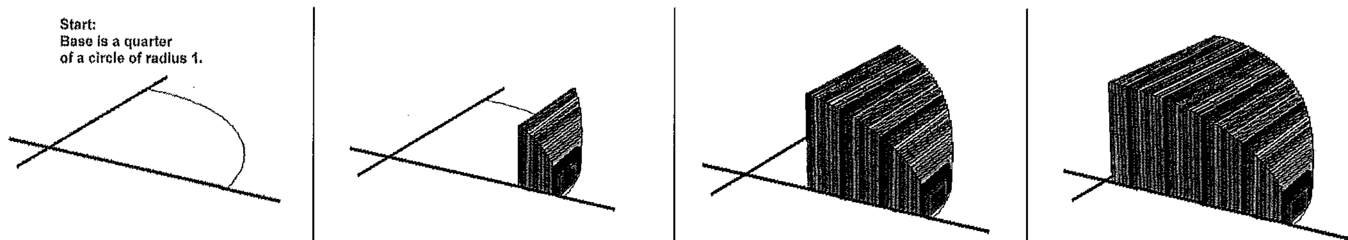

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Example 3: Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are semicircles whose base are perpendicular to the y-axis.

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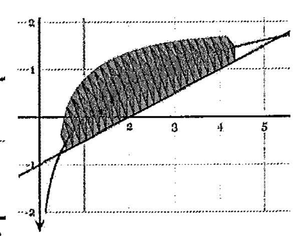
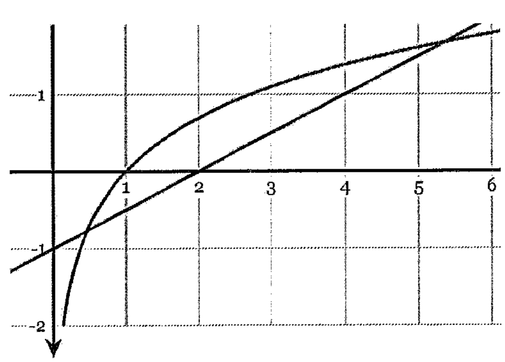
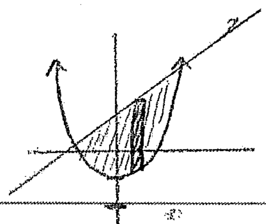




7.2c Homework p. 465-466 #61, 62, 63

* Volume of Known Cross Sections

61) $y = x+1$, $y = x^2-1$, perpendicular to x-axis



a) Squares

base $= x+1-(x^2-1)$

$= \cancel{x+1-x^2+1}$

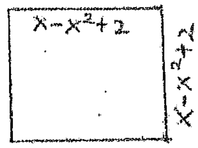
$= x-x^2+2$

*find bounds

$x^2-1 = x+1$

$x^2-x-2=0$

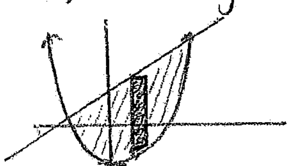
$(x-2)(x+1)=0$ $\boxed{x=2,-1}$



$$V=\int_{-1}^{2}(\text{base})^2\,dx = \int_{-1}^{2}\left[x-x^2+2\right]^2 dx$$

$$= \boxed{\frac{81}{10}\text{ units}^3}$$

b) Rectangle of height 1



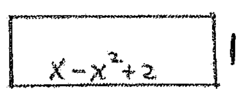
Area $=(\text{base})(1)$

base $= x+1-(x^2-1)$

$= x-x^2+2$

*bounds:

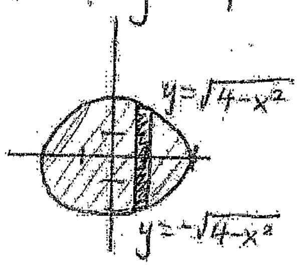
$x=-1, 2$



$$V=\int_{-1}^{2}(x^2-x+2)(1)\,dx = \boxed{\frac{9}{2}\text{ units}^3}$$

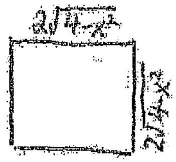
7.2c HW continued

62) $x^2+y^2=4$ perpendicular to x-axis



base $=\sqrt{4-x^2}-(-\sqrt{4-x^2})=2\sqrt{4-x^2}$

a) Squares

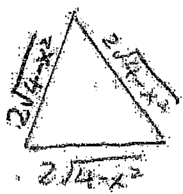


$$V=\int_{-2}^{2}\left(2\sqrt{4-x^2}\right)^2 dx$$

$$\boxed{V=\frac{128}{3}\text{ units}^3}$$

b) Equilateral triangles

base $=2\sqrt{4-x^2}$



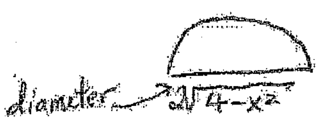
Area $=\frac{\sqrt{3}}{4}(\text{base})^2$

$$V=\frac{\sqrt{3}}{4}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2 dx$$

$$V=\frac{32\sqrt{3}}{3}\approx\boxed{18.475}$$

c) Semicircles

base $=2\sqrt{4-x^2}$



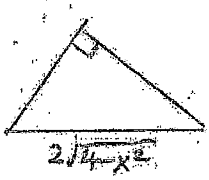
$A=\frac{\pi}{8}(\text{diameter})^2$

$$V=\frac{\pi}{8}\int_{-2}^{2}\left[2\sqrt{4-x^2}\right]^2 dx$$

$$\boxed{V=\frac{16}{3}\pi}$$

d) Isosceles right triangles

base $=2\sqrt{4-x^2}$



hypotenuse $=2\sqrt{4-x^2}$

$A=\frac{1}{4}(\text{hypotenuse})^2$

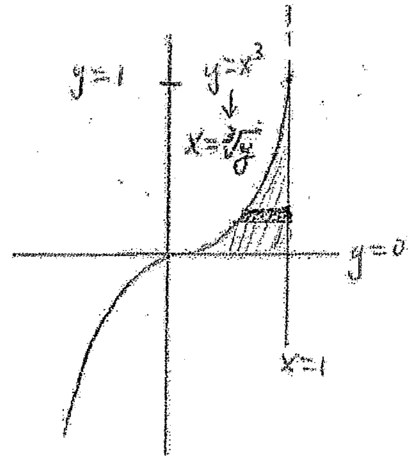
$=\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$

$$V=\int_{-2}^{2}\frac{1}{4}\left(2\sqrt{4-x^2}\right)^2$$

$$\boxed{V=\frac{32}{3}\text{ units}^3}$$

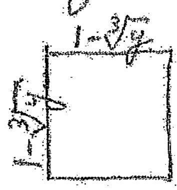
7.2c HW (continued)

63) graph bounded by $y=x^3$, $y=0$, $x=1$
(perpendicular to the y-axis)



base $= 1-\sqrt[3]{y}$ | *find bounds $\sqrt[3]{y}=1$, $y=1$

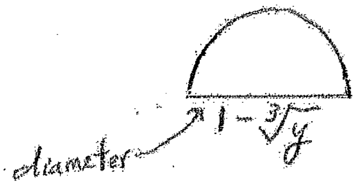
a) squares



$$V=\int_0^1 \left(1-\sqrt[3]{y}\right)^2 dy$$

$$\boxed{V=\frac{1}{10}}$$

b) semicircle

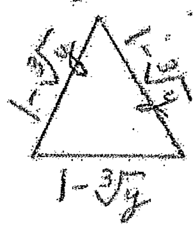


$$A=\frac{\pi}{8}(\text{diameter})^2$$

$$V=\frac{\pi}{8}\int_0^1\left[1-\sqrt[3]{y}\right]^2$$

$$V=\frac{\pi}{8}\left(\frac{1}{10}\right)=\boxed{\frac{\pi}{80}\text{ units}^3}$$

c) Equilateral triangles



$$A=\frac{\sqrt{3}}{4}(\text{base})^2$$

$$V=\frac{\sqrt{3}}{4}\int_0^1\left[1-\sqrt[3]{y}\right]^2 dy$$

$$V=\frac{\sqrt{3}}{4}\left(\frac{1}{10}\right)$$

$$=\boxed{\frac{\sqrt{3}}{40}\text{ units}^3}$$