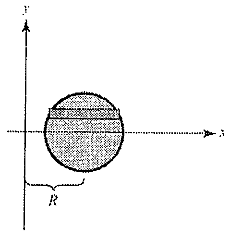
76. (a) $(x - R)^2 + y^2 = r^2$

$$x = R \pm \sqrt{r^2 - y^2}$$

$$V = 2\pi \int_0^r \left(\left[R + \sqrt{r^2 - y^2}\right]^2 - \left[R - \sqrt{r^2 - y^2}\right]^2\right) dy = 2\pi \int_0^r 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$$



(b) $\int_0^r \sqrt{r^2 - y^2}\, dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

$$V = 8\pi R\left(\tfrac{1}{4}\pi r^2\right) = 2\pi^2 r^2 R$$

Section 7.3 Volume: The Shell Method

1. $p(x) = x,\ h(x) = x$

$$V = 2\pi \int_0^2 x(x)\, dx = \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

2. $p(x) = x,\ h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x)\, dx$$

$$= 2\pi \int_0^1 \left(x - x^2\right) dx = 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

3. $p(x) = x,\ h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx = \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

4. $p(x) = x,\ h(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$

$$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx$$

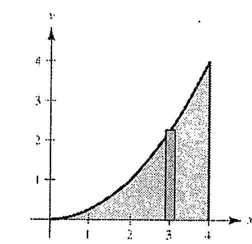
$$= 2\pi\left[x^2 - \frac{x^4}{8}\right]_0^2 = 2\pi(4 - 2) = 4\pi$$

5. $p(x) = x,\ h(x) = \frac{1}{4}x^2$

$$V = 2\pi \int_0^4 x\left(\frac{1}{4}x^2\right) dx$$

$$= \frac{\pi}{2}\left[\frac{x^4}{4}\right]_0^4$$

$$= 32\pi$$

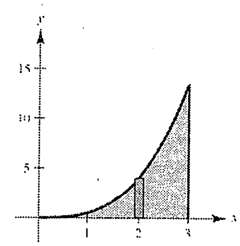


6. $p(x) = x,\ h(x) = \frac{1}{2}x^3$

$$V = 2\pi \int_0^3 x\left(\frac{x^3}{2}\right) dx$$

$$= \pi\left[\frac{x^5}{5}\right]_0^3$$

$$= \frac{243\pi}{5}$$

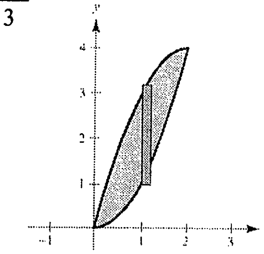


7. $p(x) = x,\ h(x) = \left(4x - x^2\right) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x\left(4x - 2x^2\right) dx$$

$$= 4\pi \int_0^2 \left(2x^2 - x^3\right) dx$$

$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



76. (a) $(x - R)^2 + y^2 = r^2$

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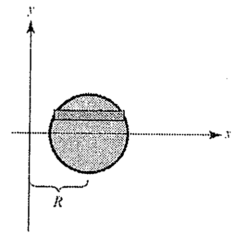
$$V = 2\pi \int_0^r \left(\left[R + \sqrt{r^2 - y^2}\right]^2 - \left[R - \sqrt{r^2 - y^2}\right]^2\right) dy = 2\pi \int_0^r 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$$



(b) $\int_0^r \sqrt{r^2 - y^2}\, dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

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Section 7.3 Volume: The Shell Method

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2. $p(x) = x$, $h(x) = 1 - x$

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$$= 2\pi \int_0^1 \left(x - x^2\right) dx = 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

3. $p(x) = x$, $h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx = \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

4. $p(x) = x$, $h(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$

$$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx$$

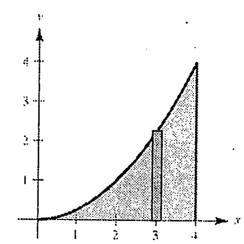
$$= 2\pi\left[x^2 - \frac{x^4}{8}\right]_0^2 = 2\pi(4 - 2) = 4\pi$$

5. $p(x) = x$, $h(x) = \frac{1}{4}x^2$

$$V = 2\pi \int_0^4 x\left(\frac{1}{4}x^2\right) dx$$

$$= \frac{\pi}{2}\left[\frac{x^4}{4}\right]_0^4$$

$$= 32\pi$$

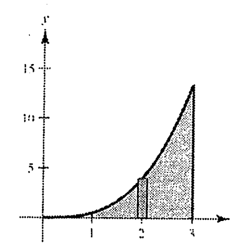


6. $p(x) = x$, $h(x) = \frac{1}{2}x^3$

$$V = 2\pi \int_0^3 x\left(\frac{x^3}{2}\right) dx$$

$$= \pi\left[\frac{x^5}{5}\right]_0^3$$

$$= \frac{243\pi}{5}$$

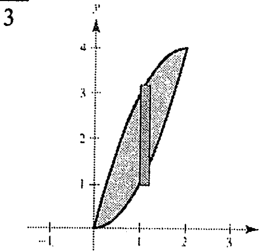


7. $p(x) = x$, $h(x) = \left(4x - x^2\right) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x\left(4x - 2x^2\right) dx$$

$$= 4\pi \int_0^2 \left(2x^2 - x^3\right) dx$$

$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$

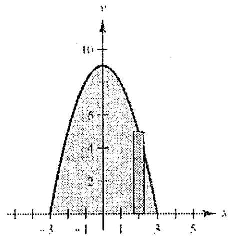


8. $p(x) = x,\ h(x) = 9 - x^2$

$$V = 2\pi \int_0^3 x(9 - x^2)\,dx$$

$$= 2\pi\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$$

$$= 2\pi\left[\frac{81}{2} - \frac{81}{4}\right] = \frac{81\pi}{2}$$



9. $p(x) = x$

$$h(x) = 4 - (4x - x^2)$$

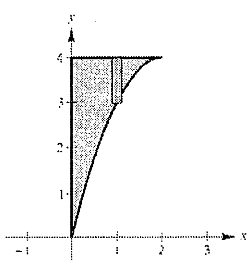
$$= x^2 - 4x + 4$$

$$V = 2\pi \int_0^2 x(x^2 - 4x + 4)\,dx$$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$

$$= \frac{8\pi}{3}$$

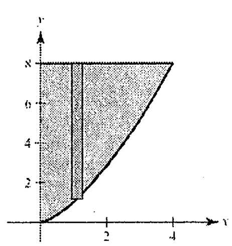


10. $p(x) = x,\ h(x) = 8 - x^{3/2}$

$$V = 2\pi \int_0^4 x(8 - x^{3/2})\,dx$$

$$= 2\pi\left[4x^2 - \frac{2}{7}x^{7/2}\right]_0^4$$

$$= 2\pi\left[64 - \frac{2}{7}(128)\right] = \frac{384\pi}{7}$$



11. $p(x) = x,\ h(x) = \sqrt{x - 2}$

$$V = 2\pi \int_2^4 x\sqrt{x - 2}\,dx$$

Let $u = x - 2$, $x = u + 2$, $du = dx$.

When $x = 2$, $u = 0$.

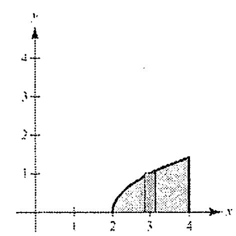
When $x = 4$, $u = 2$.

$$V = 2\pi \int_0^2 (u + 2)u^{1/2}\,du$$

$$= 2\pi\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^2$$

$$= 2\pi\left[\frac{2}{5}(2)^{5/2} + \frac{4}{3}(2)^{3/2}\right]$$

$$= 2\pi\sqrt{2}\left[\frac{2}{5}(4) + \frac{4}{3}(2)\right] = \frac{128\sqrt{2}\pi}{15}$$

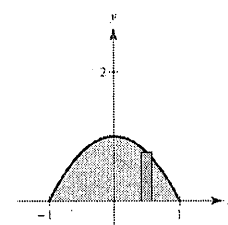


12. $p(x) = x,\ h(x) = 1 - x^2$

$$V = 2\pi \int_0^1 x(1 - x^2)\,dx$$

$$= 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$



13. $p(x) = x,\ h(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

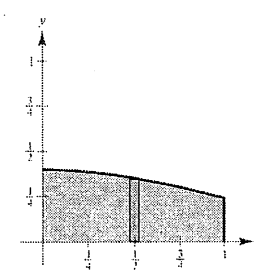
$$V = 2\pi \int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$$

$$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$$

$$= \left[-\sqrt{2\pi}e^{-x^2/2}\right]_0^1$$

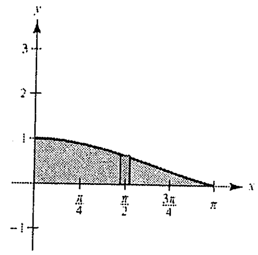
$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$

$$\approx 0.986$$



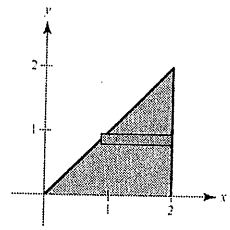
14. $p(x) = x,\ h(x) = \dfrac{\sin x}{x}$

$$V = 2\pi\int_0^{\pi} x\left[\frac{\sin x}{x}\right] dx$$
$$= 2\pi\int_0^{\pi} \sin x\, dx$$
$$= \left[-2\pi \cos x\right]_0^{\pi} = 4\pi$$



15. $p(y) = y,\ h(y) = 2 - y$

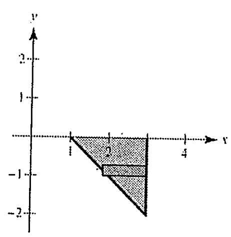
$$V = 2\pi\int_0^{2} y(2 - y)\, dy$$
$$= 2\pi\int_0^{2}\left(2y - y^2\right) dy$$
$$= 2\pi\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}$$



16. $p(y) = -y$ $\quad$ (So, $p(y) \ge 0$ on $[-2, 0]$)

$h(y) = 3 - (1 - y) = 2 + y$

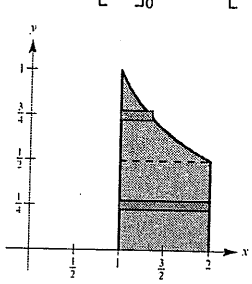
$$V = 2\pi\int_{-2}^{0}(-y)(2 + y)\, dy$$
$$= 2\pi\int_{-2}^{0}\left[-2y - y^2\right] dy$$
$$= 2\pi\left[-y^2 - \frac{y^3}{3}\right]_{-2}^{0}$$
$$= 2\pi\left[0 - \left(-4 + \frac{8}{3}\right)\right]$$
$$= 2\pi\frac{4}{3}$$
$$= \frac{8\pi}{3}$$



17. $p(y) = y$ and $h(y) = 1$ if $0 \le y < \dfrac{1}{2}$.

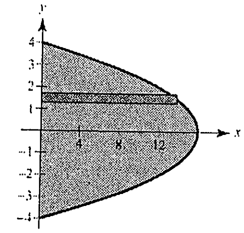
$p(y) = y$ and $h(y) = \dfrac{1}{y} - 1$ if $\dfrac{1}{2} \le y \le 1$.

$$V = 2\pi\int_0^{1/2} y\, dy + 2\pi\int_{1/2}^{1}(1 - y)\, dy$$
$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



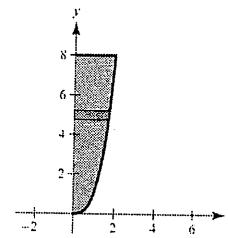
18. $p(y) = y,\ h(y) = 16 - y^2$

$$V = 2\pi\int_0^{4} y\left(16 - y^2\right) dy$$
$$= 2\pi\int_0^{4}\left(16y - y^3\right) dy$$
$$= 2\pi\left[8y^2 - \frac{y^4}{4}\right]_0^4$$
$$= 2\pi(128 - 64)$$
$$= 128\pi$$



19. $p(y) = y,\ h(y) = \sqrt[3]{y}$

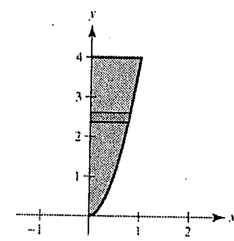
$$V = 2\pi\int_0^{8} y\sqrt[3]{y}\, dy$$
$$= 2\pi\int_0^{8} y^{4/3}\, dy$$
$$= \left[2\pi\left(\frac{3}{7}\right)y^{7/3}\right]_0^8$$
$$= \frac{6\pi}{7}\left(2^7\right) = \frac{768\pi}{7}$$



20. $y = 4x^2,\ x = \dfrac{\sqrt{y}}{2}$

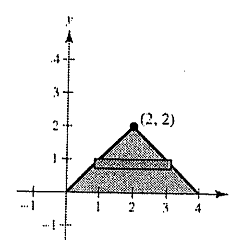
$p(y) = y,\ h(y) = \dfrac{\sqrt{y}}{2}$

$$V = 2\pi\int_0^{4} y\left(\frac{\sqrt{y}}{2}\right) dy$$
$$= \pi\int_0^{4} y^{3/2}\, dy$$
$$= \pi\left[\frac{2}{5}y^{5/2}\right]_0^4$$
$$= \frac{64\pi}{5}$$



21. $p(y) = y,\ h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi\int_0^{2} y(4 - 2y)\, dy$$
$$= 2\pi\int_0^{2}\left(4y - 2y^2\right) dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$
$$= 2\pi\left(8 - \frac{16}{3}\right) = \frac{16\pi}{3}$$



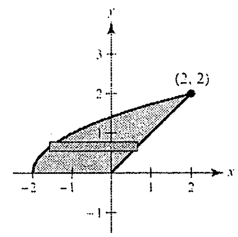
22. $p(y) = y,\ h(y) = y - (y^2 - 2) = 2 + y - y^2$

$$V = 2\pi\int_0^2 y(2 + y - y^2)\,dy$$

$$= 2\pi\int_0^2 (2y + y^2 - y^3)\,dy$$

$$= 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi\left(4 + \frac{8}{3} - 4\right) = \frac{16\pi}{3}$$



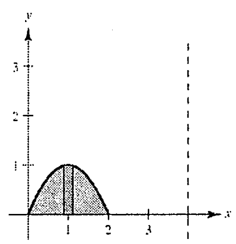
23. $p(x) = 4 - x,\ h(x) = 2x - x^2$

$$V = 2\pi\int_0^2 (4 - x)(2x - x^2)\,dx$$

$$= 2\pi\int_0^2 (8x - 6x^2 + x^3)\,dx$$

$$= 2\pi\left[4x^2 - 2x^3 + \frac{x^4}{4}\right]_0^2$$

$$= 2\pi[16 - 16 + 4] = 8\pi$$

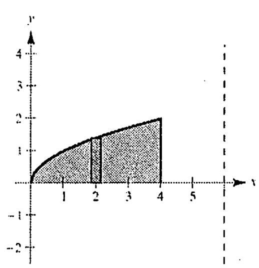


24. $p(x) = 6 - x,\ h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 (6 - x)\sqrt{x}\,dx$$

$$= 2\pi\int_0^4 (6x^{1/2} - x^{3/2})\,dx$$

$$= 2\pi\left[4x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = \frac{192\pi}{5}$$

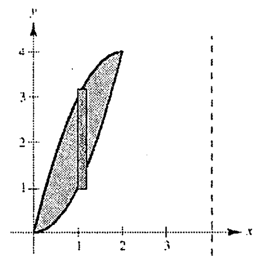


25. $p(x) = 4 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (4 - x)(4x - 2x^2)\,dx$$

$$= 2\pi(2)\int_0^2 (x^3 - 6x^2 + 8x)\,dx$$

$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$



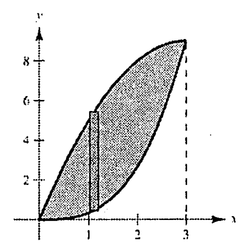
26. $p(x) = 3 - x,\ h(x) = (6x - x^2) - \frac{1}{3}x^3$

$$V = 2\pi\int_0^3 (3 - x)\left(6x - x^2 - \frac{x^3}{3}\right)dx$$

$$= 2\pi\int_0^3 \left(\frac{x^4}{3} - 9x^2 + 18x\right)dx$$

$$= 2\pi\left[\frac{x^5}{15} - 3x^3 + 9x^2\right]_0^3$$

$$= \frac{162\pi}{5}$$



27. The shell method would be easier:

$$V = 2\pi\int_0^4 \left[4 - (y - 2)^2\right]y\,dy$$

Using the disk method:

$$V = \pi\int_0^4 \left[\left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2\right]dx$$

$\left[\textbf{Note: } V = \frac{128\pi}{3}\right]$

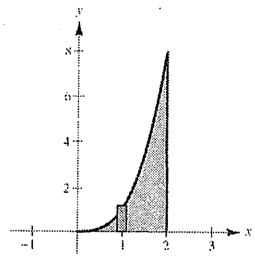
28. The shell method is easier: $V = 2\pi\int_0^{\ln 4} x(4 - e^x)\,dx$

Using the disk method, $x = \ln(4 - y)$ and $V = \pi\int_0^3 (\ln(4 - y))^2\,dy$. $\left[\textbf{Note: } V = \pi\left[8(\ln 2)^2 - 8\ln 2 + 3\right]\right]$

29. (a) **Disk**

$R(x) = x^3, r(x) = 0$

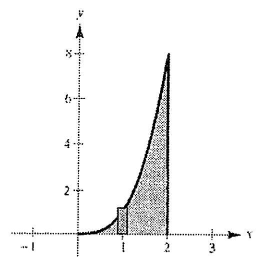
$$V = \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



(b) **Shell**

$p(x) = x, h(x) = x^3$

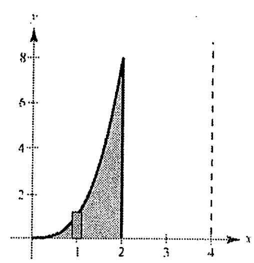
$$V = 2\pi\int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



(c) **Shell**

$p(x) = 4 - x, h(x) = x^3$

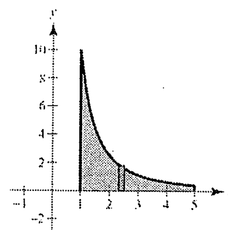
$$V = 2\pi\int_0^2 (4 - x)x^3\,dx$$
$$= 2\pi\int_0^2 \left(4x^3 - x^4\right)dx$$
$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



30. (a) **Disk**

$R(x) = \frac{10}{x^2}, r(x) = 0$

$$V = \pi\int_1^5 \left(\frac{10}{x^2}\right)^2 dx$$
$$= 100\pi\int_1^5 x^{-4}\,dx$$
$$= 100\pi\left[\frac{x^{-3}}{-3}\right]_1^5$$
$$= -\frac{100\pi}{3}\left(\frac{1}{125} - 1\right) = \frac{496}{15}\pi$$



(b) **Shell**

$R(x) = x, r(x) = 0$

$$V = 2\pi\int_1^5 x\left(\frac{10}{x^2}\right)dx$$
$$= 20\pi\int_1^5 \frac{1}{x}\,dx$$
$$= 20\pi\Big[\ln|x|\Big]_1^5 = 20\pi \ln 5$$

(c) **Disk**

$R(x) = 10, r(x) = 10 - \frac{10}{x^2}$

$$V = \pi\int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right]dx$$
$$= \pi\left[\frac{100}{3x^3} - \frac{200}{x}\right]_1^5 = \frac{1904}{15}\pi$$

31. (a) **Shell**

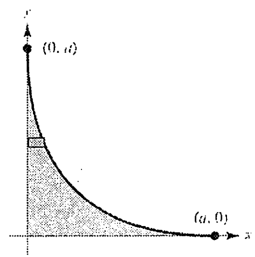
$$p(y) = y,\ h(y) = \left(a^{1/2} - y^{1/2}\right)^2$$

$$V = 2\pi \int_0^a y\left(a - 2a^{1/2}y^{1/2} + y\right) dy$$

$$= 2\pi \int_0^a \left(ay - 2a^{1/2}y^{3/2} + y^2\right) dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left(\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right) = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

(c) **Shell**

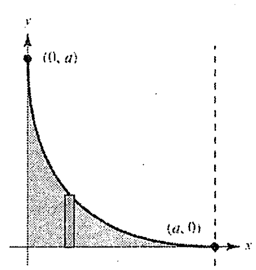
$$p(x) = a - x,\ h(x) = \left(a^{1/2} - x^{1/2}\right)^2$$

$$V = 2\pi \int_0^a (a - x)\left(a^{1/2} - x^{1/2}\right)^2 dx$$

$$= 2\pi \int_0^a \left(a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4\pi a^3}{15}$$



32. (a) **Disk**

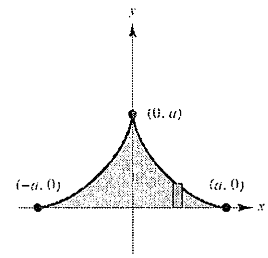
$$R(x) = \left(a^{2/3} - x^{2/3}\right)^{3/2},\ r(x) = 0$$

$$V = \pi \int_{-a}^a \left(a^{2/3} - x^{2/3}\right)^3 dx$$

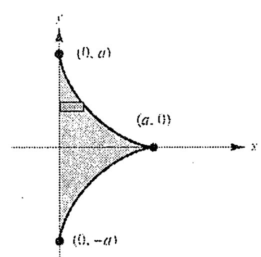
$$= 2\pi \int_0^a \left(a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a$$

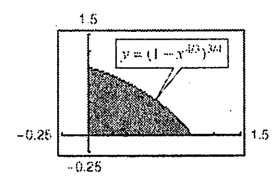
$$= 2\pi\left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3\right) = \frac{32\pi a^3}{105}$$



(b) Same as part (a) by symmetry



33. (a)

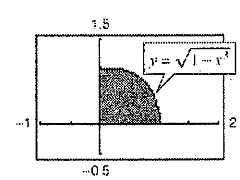


(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$$y = \left(1 - x^{4/3}\right)^{3/4}$$

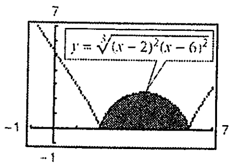
$$V = 2\pi \int_0^1 x\left(1 - x^{4/3}\right)^{3/4} dx \approx 1.5056$$

34. (a)



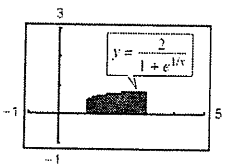
(b) $V = 2\pi \int_0^1 x\sqrt{1 - x^3}\, dx \approx 2.3222$

35. (a)



(b) $V = 2\pi\int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\,dx \approx 187.249$

36. (a)



(b) $V = 2\pi\int_1^3 \dfrac{2x}{1+e^{1/x}}\,dx \approx 19.0162$

37. Answers will vary.

(a) The rectangles would be vertical.

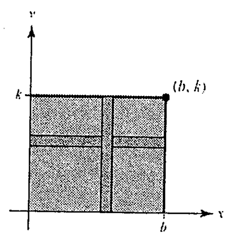
(b) The rectangles would be horizontal.

38. (a) radius $= k$

height $= b$

(b) radius $= b$

height $= k$

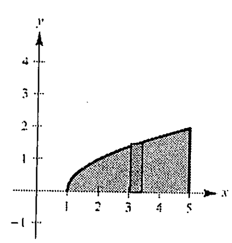


39. $\pi\int_1^5 (x-1)\,dx = \pi\int_1^5 \left(\sqrt{x-1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ about the x-axis by using the disk method.

$2\pi\int_0^2 y\left[5 - \left(y^2+1\right)\right] dy$

represents this same volume by using the shell method.



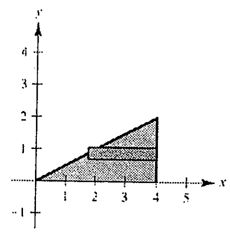
Disk method

40. $2\pi\int_0^4 x\left(\dfrac{x}{2}\right) dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the y-axis by using the shell method.

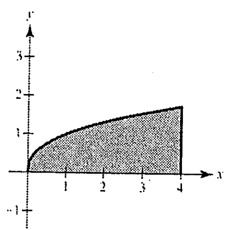
$\pi\int_0^2 \left[16 - (2y)^2\right] dy = \pi\int_0^2 \left[(4)^2 - (2y)^2\right] dy$

represents this same volume by using the disk method.



Disk method

41.



(a) Around x-axis: $V = \pi\int_0^4 \left(x^{2/5}\right)^2 dx = \left[\pi\dfrac{5}{9}x^{9/5}\right]_0^4$

$= \dfrac{5}{9}\pi(4)^{9/5} \approx 6.7365\pi$

(b) Around y-axis: $V = 2\pi\int_0^4 x\left(x^{2/5}\right) dx$

$= \left[2\pi\dfrac{5}{12}x^{12/5}\right]_0^4 \approx 23.2147\pi$

(c) Around $x = 4$:

$V = 2\pi\int_0^4 (4-x)x^{2/5}\,dx \approx 16.5819\pi$

So, $(a) < (c) < (b)$.

42. (a) The figure will be a circle of radius AB and center A.

(b) The figure will be a circular cylinder of radius AB.

(c) Disk method: $V = \pi\int_0^3 \left[g(y)\right]^2 dy$

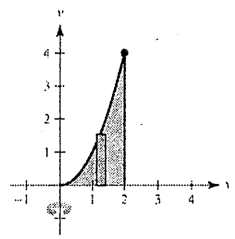
Shell method: $V = 2\pi\int_0^{2.45} x\,f(x)\,dx$

43. $2\pi \int_0^2 x^3\,dx = 2\pi \int_0^2 x(x^2)\,dx$

(a) Plane region bounded by

$y = x^2, y = 0, x = 0, x = 2$

(b) Revolved about the y-axis

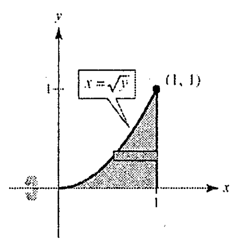


Other answers possible.

44. $2\pi \int_0^1 (y - y^{3/2})\,dy = 2\pi \int_0^1 y(1 - \sqrt{y})\,dy$

(a) Plane region bounded by $x = \sqrt{y}, x = 1, y = 0$

(b) Revolved about the x-axis



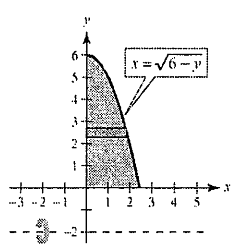
Other answers possible.

45. $2\pi \int_0^6 (y + 2)\sqrt{6 - y}\,dy$

(a) Plane region bounded by

$x = \sqrt{6 - y}, x = 0, y = 0$

(b) Revolved around line $y = -2$



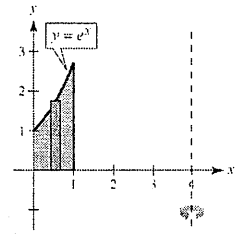
Other answers possible.

46. $2\pi \int_0^1 (4 - x)e^x\,dx$

(a) Plane region bounded by

$y = e^x, y = 0, x = 0, x = 1$

(b) Revolved about the line $x = 4$



47. $p(x) = x, h(x) = 2 - \frac{1}{2}x^2$

$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx$

$= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx$

$= 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi$ (total volume)

Now find x_0 such that:

$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

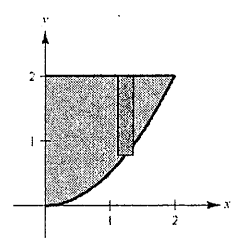
$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, because the other root is too large.

Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$



48. Total volume of the hemisphere is

$\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method,

$p(x) = x, h(x) = \sqrt{9 - x^2}$. Find x_0 such that:

$$6\pi = 2\pi \int_0^{x_0} x\sqrt{9 - x^2}\, dx$$

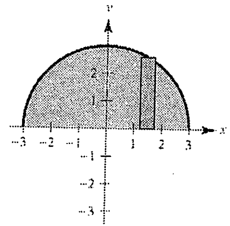
$$6 = -\int_0^{x_0} (9 - x^2)^{1/2}(-2x)\, dx$$

$$= \left[-\frac{2}{3}(9 - x^2)^{3/2}\right]_0^{x_0} = 18 - \frac{2}{3}(9 - x_0^2)^{3/2}$$

$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460$$

Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$



49. $V = 4\pi \int_{-1}^{1} (2 - x)\sqrt{1 - x^2}\, dx$

$$= 8\pi \int_{-1}^{1} \sqrt{1 - x^2}\, dx - 4\pi \int_{-1}^{1} x\sqrt{1 - x^2}\, dx$$

$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^{1} x(1 - x^2)^{1/2}(-2)\, dx$$

$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

50. $V = 4\pi \int_{-r}^{r} (R - x)\sqrt{r^2 - x^2}\, dx$

$$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx - 4\pi \int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx$$

$$= 4\pi R\left(\frac{\pi r^2}{2}\right) + \left[2\pi\left(\frac{2}{3}\right)(r^2 - x^2)^{3/2}\right]_{-r}^{r}$$

$$= 2\pi^2 r^2 R$$

51. (a) $\frac{d}{dx}[\sin x - x\cos x + C] = \cos x + x\sin x - \cos x$

$$= x\sin x$$

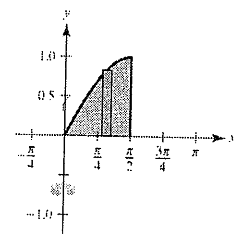
So, $\int x\sin x\, dx = \sin x - x\cos x + C$.

(b) (i) $p(x) = x, h(x) = \sin x$

$$V = 2\pi \int_0^{\pi/2} x\sin x\, dx$$

$$= 2\pi[\sin x - x\cos x]_0^{\pi/2}$$

$$= 2\pi[(1 - 0) - 0] = 2\pi$$



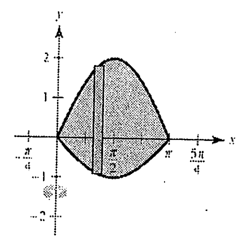
(ii) $p(x) = x, h(x) = 2\sin x - (-\sin x) = 3\sin x$

$$V = 2\pi \int_0^{\pi} x(3\sin x)\, dx$$

$$= 6\pi \int_0^{\pi} x\sin x\, dx$$

$$= 6\pi[\sin x - x\cos x]_0^{\pi}$$

$$= 6\pi(\pi) = 6\pi^2$$



52. (a) $\frac{d}{dx}[\cos x + x \sin x + C] = -\sin x + \sin x + x \cos x$

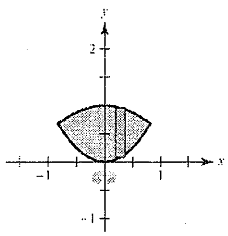
$= x \cos x$

Hence, $\int x \cos x\, dx = \cos x + x \sin x + C.$

(b) (i) $x^2 = \cos x \Rightarrow x \approx \pm 0.8241$

$$V \approx 2(2\pi)\int_0^{0.8241} x\left[\cos x - x^2\right] dx$$

$$= 4\pi\left[\cos x + x \sin x - \frac{x^4}{4}\right]_0^{0.8241} \approx 2.1205$$

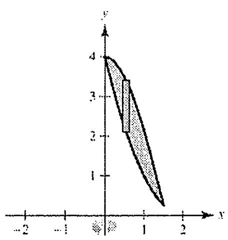


(ii) $4 \cos x = (x - 2)^2 \Rightarrow x = 0, 1.5110$

$$V \approx 2\pi\int_0^{1.511} x\left[4\cos x - (x-2)^2\right] dx$$

$$= 2\pi\int_0^{1.511}\left[4\cos x + 4x\sin x - \frac{(x-2)^3}{3}\right]_0^{1.511}$$

$= 6.2993$



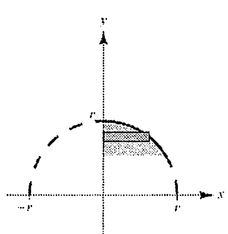
53. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

$$V = \pi\int_{r-h}^{r}\left(r^2 - y^2\right) dy$$

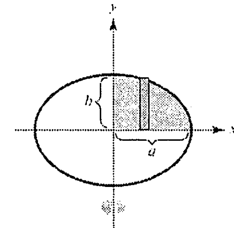
$$= \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$



54. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$



$p(x) = x,\ h(x) = b\sqrt{1 - \frac{x^2}{a^2}}$

$$V = 2(2\pi)\int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\, dx$$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\, x\, dx$$

$$= \frac{4\pi b}{a}\left[\left(\frac{-\left(a^2 - x^2\right)^{3/2}}{3}\right)\right]_0^a$$

$$= \frac{4\pi b}{3a}a^3 = \frac{4}{3}\pi a^2 b$$

If the region is revolved about the x-axis, then by symmetry the volume would be $V = \frac{4}{3}\pi ab^2$.

Note: If $a = b$, then volume is that of a sphere.

55. (a) Area of region $= \int_0^b \left[ab^n - ax^n\right] dx$

$$= \left[ab^n x - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right)$$

$$= ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}\left[n/(n+1)\right]}{\left(ab^n\right)b} = \frac{n}{n+1}$$

(b) $\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\lim_{n\to\infty}\left(ab^n\right)b = \infty$

(c) **Disk Method:**

$$V = 2\pi\int_0^b x\left(ab^n - ax^n\right) dx$$

$$= 2\pi a\int_0^b \left(xb^n - x^{n+1}\right) dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}\left[n/(n+2)\right]}{\left(\pi b^2\right)\left(ab^n\right)} = \left(\frac{n}{n+2}\right)$$

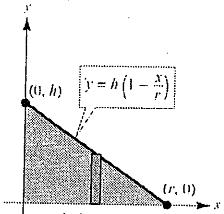
(d) $\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = 1$

$\lim_{n\to\infty}\left(\pi b^2\right)\left(ab^n\right) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = b$.

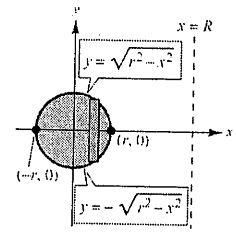
56. (a) $2\pi\int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h.



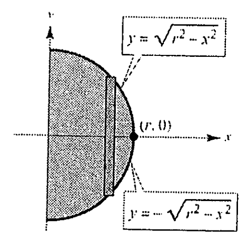
(b) $2\pi\int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



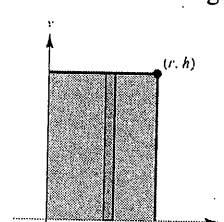
(c) $2\pi\int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii)

is the volume of a sphere with radius r.



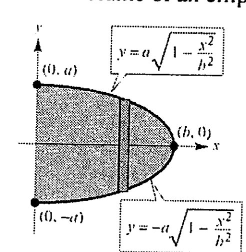
(d) $2\pi\int_0^r hx\, dx$ (i)

is the volume of a right circular cylinder with a radius of r and a height of h.



(e) $2\pi\int_0^b 2ax\sqrt{1 - \left(x^2/b^2\right)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



57. (a) $V = 2\pi \int_0^4 xf(x)\,dx = \frac{2\pi(40)}{3(4)}\left[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0\right] = \frac{20\pi}{3}(5800) \approx 121{,}475 \text{ ft}^3$

(b) Top line: $y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x + 50$

Bottom line: $y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \Rightarrow y = -2x + 80$

$$V = 2\pi \int_0^{20} x\left(-\frac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80)\,dx$$

$$= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} \left(-2x^2 + 80x\right) dx$$

$$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40} = 2\pi\left(\frac{26{,}000}{3}\right) + 2\pi\left(\frac{32{,}000}{3}\right) \approx 121{,}475 \text{ ft}^3$$

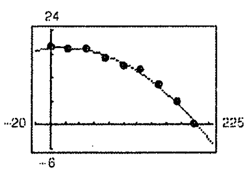
(Note that Simpson's Rule is exact for this problem.)

58. (a) $V = 2\pi \int_0^{200} xf(x)\,dx$

$$\approx \frac{2\pi(200)}{3(8)}\left[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0\right]$$

$$\approx 1{,}366{,}593 \text{ ft}^3$$

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800) = 1{,}343{,}345 \text{ ft}^3$

(d) Number of gallons $\approx V(7.48) = 10{,}048{,}221$ gal

59. $V_1 = \pi \int_{1/4}^{c} \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_{1/4}^{c} = \pi\left[-\frac{1}{c} + 4\right] = \frac{4c - 1}{c}\pi$

$$V_2 = \left[2\pi \int_{1/4}^{c} x\left(\frac{1}{x}\right) dx = 2\pi x\right]_{1/4}^{c} = 2\pi\left(c - \frac{1}{4}\right)$$

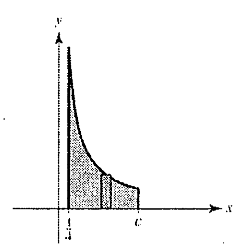
$$V_1 = V_2 \Rightarrow \frac{4c - 1}{c}\pi = 2\pi\left(c - \frac{1}{4}\right)$$

$$4c - 1 = 2c\left(c - \frac{1}{4}\right)$$

$$4c^2 - 9c + 2 = 0$$

$$(4c - 1)(c - 2) = 0$$

$$c = 2 \quad \left(c = \frac{1}{4} \text{ yields no volume.}\right)$$



60. (a) $p(x) = x,\ h(x) = r^2 - x^2$

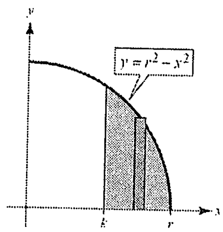
Shell method:

$$V = 2\pi\int_k^r x\left(r^2 - x^2\right)dx$$

$$= -\pi\int_k^r \left(r^2 - x^2\right)(-2x)\,dx$$

$$= -\pi\left[\frac{\left(r^2 - x^2\right)^2}{2}\right]_k^r$$

$$= -\pi\left[0 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$



(b) $y = r^2 - x^2$

$x = \sqrt{r^2 - y}$

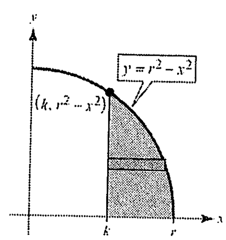
Disk method:

$$V = \pi\int_0^{r^2-k^2}\left[\left(\sqrt{r^2 - y}\right)^2 - k^2\right]dy$$

$$= \pi\int_0^{r^2-k^2}\left[r^2 - y - k^2\right]dy$$

$$= \pi\left[\left(r^2 - k^2\right)y - \frac{y^2}{2}\right]_0^{r^2-k^2}$$

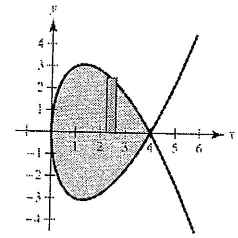
$$= \pi\left[\left(r^2 - k^2\right)^2 - \frac{\left(r^2 - k^2\right)^2}{2}\right] = \frac{\pi}{2}\left(r^2 - k^2\right)^2$$



61. $y^2 = x(4 - x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4 - x)^2} = (4 - x)\sqrt{x}$

$y_2 = -\sqrt{x(4 - x)^2} = -(4 - x)\sqrt{x}$



(a) $V = \pi\int_0^4 x(4 - x)^2\,dx$

$$= \pi\int_0^4\left(x^3 - 8x^2 + 16x\right)dx$$

$$= \pi\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$$

(b) $V = 4\pi\int_0^4 x(4 - x)\sqrt{x}\,dx$

$$= 4\pi\int_0^4\left(4x^{3/2} - x^{5/2}\right)dx$$

$$= 4\pi\left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$$

(c) $V = 4\pi\int_0^4 (4 - x)(4 - x)\sqrt{x}\,dx$

$$= 4\pi\int_0^4\left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right)dx$$

$$= 4\pi\left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$$

Section 7.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (8, 15)$

(a) $d = \sqrt{(8 - 0)^2 + (15 - 0)^2}$

$= \sqrt{64 + 225}$

$= \sqrt{289} = 17$

(b) $y = \frac{15}{8}x$

$y' = \frac{15}{8}$

$$s = \int_0^8\sqrt{1 + \left(\frac{15}{8}\right)^2}\,dx = \int_0^8 \frac{17}{8}\,dx = \left[\frac{17}{8}x\right]_0^8 = 17$$