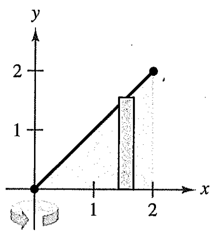
7.3 Exercises

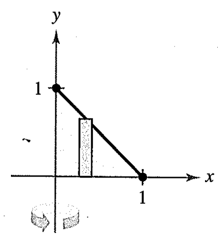
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Volume of a Solid In Exercises 1–14, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-axis.

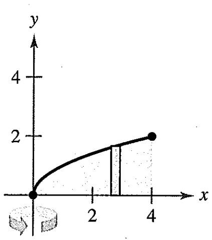
1. $y = x$



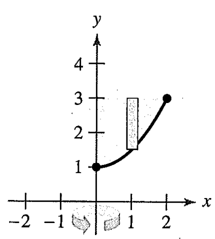
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = \frac{1}{2}x^2 + 1$



5. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$

6. $y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$

7. $y = x^2, \quad y = 4x - x^2$

8. $y = 9 - x^2, \quad y = 0$

9. $y = 4x - x^2, \quad x = 0, \quad y = 4$

10. $y = x^{3/2}, \quad y = 8, \quad x = 0$

11. $y = \sqrt{x - 2}, \quad y = 0, \quad x = 4$

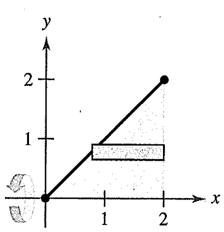
12. $y = -x^2 + 1, \quad y = 0$

13. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

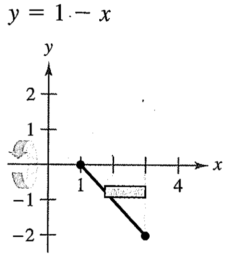
14. $y = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = \pi$

Finding the Volume of a Solid In Exercises 15–22, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the x-axis.

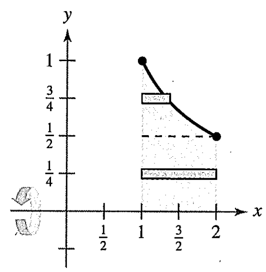
15. $y = x$



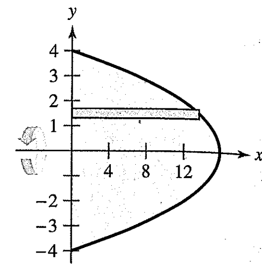
16. $y = 1 - x$



17. $y = \frac{1}{x}$



18. $x + y^2 = 16$



19. $y = x^3, \quad x = 0, \quad y = 8$

20. $y = 4x^2, \quad x = 0, \quad y = 4$

21. $x + y = 4, \quad y = x, \quad y = 0$

22. $y = \sqrt{x + 2}, \quad y = x, \quad y = 0$

Finding the Volume of a Solid In Exercises 23–26, use the shell method to find the volume of the solid generated by revolving the plane region about the given line.

23. $y = 2x - x^2, \quad y = 0,$ about the line $x = 4$

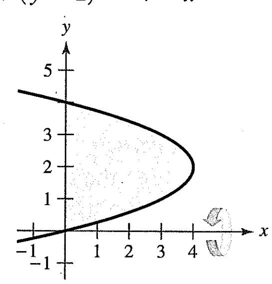
24. $y = \sqrt{x}, \quad y = 0, \quad x = 4,$ about the line $x = 6$

25. $y = x^2, \quad y = 4x - x^2,$ about the line $x = 4$

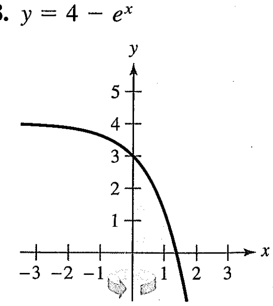
26. $y = \frac{1}{3}x^3, \quad y = 6x - x^2,$ about the line $x = 3$

Choosing a Method In Exercises 27 and 28, decide whether it is more convenient to use the disk method or the shell method to find the volume of the solid of revolution. Explain your reasoning. (Do not find the volume.)

27. $(y - 2)^2 = 4 - x$



28. $y = 4 - e^x$



Choosing a Method In Exercises 29–32, use the disk method *or* the shell method to find the volumes of the solids generated by revolving the region bounded by the graphs of the equations about the given lines.

29. $y = x^3, \quad y = 0, \quad x = 2$

(a) the x-axis (b) the y-axis (c) the line $x = 4$

30. $y = \frac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$

(a) the x-axis (b) the y-axis (c) the line $y = 10$

31. $x^{1/2} + y^{1/2} = a^{1/2}, \quad x = 0, \quad y = 0$

(a) the x-axis (b) the y-axis (c) the line $x = a$

32. $x^{2/3} + y^{2/3} = a^{2/3}, \quad a > 0$ (hypocycloid)

(a) the x-axis (b) the y-axis

Finding the Volume of a Solid **In Exercises 33–36, (a) use a graphing utility to graph the plane region bounded by the graphs of the equations, and (b) use the integration capabilities of the graphing utility to approximate the volume of the solid generated by revolving the region about the y-axis.**

33. $x^{4/3} + y^{4/3} = 1, \quad x = 0, \quad y = 0,$ first quadrant

34. $y = \sqrt{1 - x^3}, \quad y = 0, \quad x = 0$

35. $y = \sqrt[3]{(x-2)^2(x-6)^2}, \quad y = 0, \quad x = 2, \quad x = 6$

36. $y = \dfrac{2}{1 + e^{1/x}}, \quad y = 0, \quad x = 1, \quad x = 3$

WRITING ABOUT CONCEPTS

37. **Representative Rectangles** Consider a solid that is generated by revolving a plane region about the y-axis. Describe the position of a representative rectangle when using (a) the shell method and (b) the disk method to find the volume of the solid.

38. **Describing Cylindrical Shells** Consider the plane region bounded by the graphs of

$$y = k, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = b$$

where $k > 0$ and $b > 0$. What are the heights and radii of the cylinders generated when this region is revolved about (a) the x-axis and (b) the y-axis?

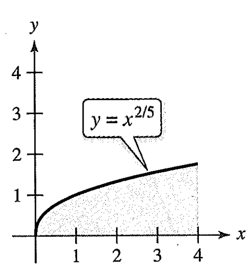
Comparing Integrals **In Exercises 39 and 40, give a geometric argument that explains why the integrals have equal values.**

39. $\pi \displaystyle\int_1^5 (x - 1)\,dx = 2\pi \int_0^2 y[5 - (y^2 + 1)]\,dy$

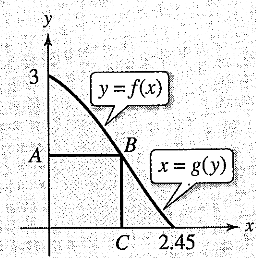
40. $\pi \displaystyle\int_0^2 [16 - (2y)^2]\,dy = 2\pi \int_0^4 x\left(\frac{x}{2}\right)dx$

41. **Comparing Volumes** The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.

(a) x-axis (b) y-axis (c) $x = 4$



42. **HOW DO YOU SEE IT?** Use the graph to answer the following.



(a) Describe the figure generated by revolving segment AB about the y-axis.

(b) Describe the figure generated by revolving segment BC about the y-axis.

(c) Assume the curve in the figure can be described as $y = f(x)$ or $x = g(y)$. A solid is generated by revolving the region bounded by the curve, $y = 0$, and $x = 0$ about the y-axis. Set up integrals to find the volume of this solid using the disk method and the shell method. (Do not integrate.)

Analyzing an Integral **In Exercises 43–46, the integral represents the volume of a solid of revolution. Identify (a) the plane region that is revolved and (b) the axis of revolution.**

43. $2\pi \displaystyle\int_0^2 x^3\,dx$

44. $2\pi \displaystyle\int_0^1 y - y^{3/2}\,dy$

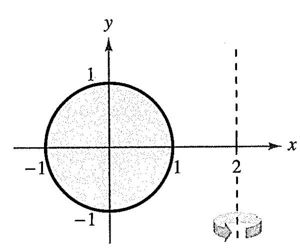
45. $2\pi \displaystyle\int_0^6 (y + 2)\sqrt{6 - y}\,dy$

46. $2\pi \displaystyle\int_0^1 (4 - x)e^x\,dx$

47. **Machine Part** A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-fourth of the volume is removed. Find the diameter of the hole.

48. **Machine Part** A solid is generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-third of the volume is removed. Find the diameter of the hole.

49. **Volume of a Torus** A torus is formed by revolving the region bounded by the circle $x^2 + y^2 = 1$ about the line $x = 2$ (see figure). Find the volume of this "doughnut-shaped" solid. (*Hint:* The integral $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$ represents the area of a semicircle.)



50. Volume of a Torus Repeat Exercise 49 for a torus formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$, where $r < R$.

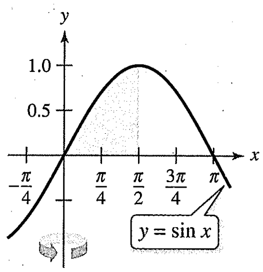
51. Finding Volumes of Solids

(a) Use differentiation to verify that

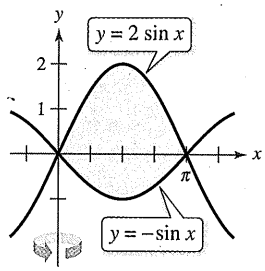
$$\int x \sin x \, dx = \sin x - x \cos x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis.

(i)



(ii)



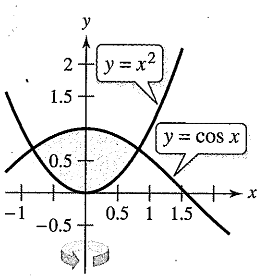
52. Finding Volumes of Solids

(a) Use differentiation to verify that

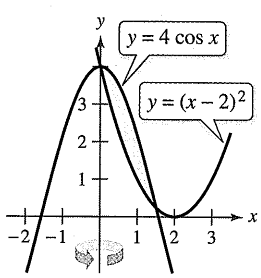
$$\int x \cos x \, dx = \cos x + x \sin x + C.$$

(b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis. (*Hint:* Begin by approximating the points of intersection.)

(i)



(ii)



53. Volume of a Segment of a Sphere Let a sphere of radius r be cut by a plane, thereby forming a segment of height h. Show that the volume of this segment is

$$\frac{1}{3}\pi h^2(3r - h).$$

54. Volume of an Ellipsoid Consider the plane region bounded by the graph of

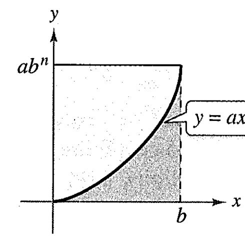
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

where $a > 0$ and $b > 0$. Show that the volume of the ellipsoid formed when this region is revolved about the y-axis is

$$\frac{4}{3}\pi a^2 b.$$

What is the volume when the region is revolved about the x-axis?

55. Exploration Consider the region bounded by the graphs of $y = ax^n$, $y = ab^n$, and $x = 0$ (see figure).



(a) Find the ratio $R_1(n)$ of the area of the region to the area of the circumscribed rectangle.

(b) Find $\lim_{n\to\infty} R_1(n)$ and compare the result with the area of the circumscribed rectangle.

(c) Find the volume of the solid of revolution formed by revolving the region about the y-axis. Find the ratio $R_2(n)$ of this volume to the volume of the circumscribed right circular cylinder.

(d) Find $\lim_{n\to\infty} R_2(n)$ and compare the result with the volume of the circumscribed cylinder.

(e) Use the results of parts (b) and (d) to make a conjecture about the shape of the graph of $y = ax^n$ $(0 \le x \le b)$ as $n \to \infty$.

56. Think About It Match each integral with the solid whose volume it represents, and give the dimensions of each solid.

(a) Right circular cone (b) Torus (c) Sphere

(d) Right circular cylinder (e) Ellipsoid

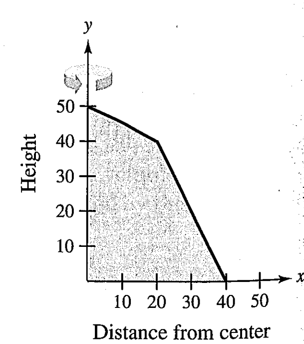
(i) $2\pi \int_0^r hx \, dx$ (ii) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$

(iii) $2\pi \int_0^r 2x\sqrt{r^2 - x^2} \, dx$ (iv) $2\pi \int_0^b 2ax \sqrt{1 - \frac{x^2}{b^2}} \, dx$

(v) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$

57. Volume of a Storage Shed A storage shed has a circular base of diameter 80 feet. Starting at the center, the interior height is measured every 10 feet and recorded in the table (see figure).

x	Height
0	50
10	45
20	40
30	20
40	0



(a) Use Simpson's Rule to approximate the volume of the shed.

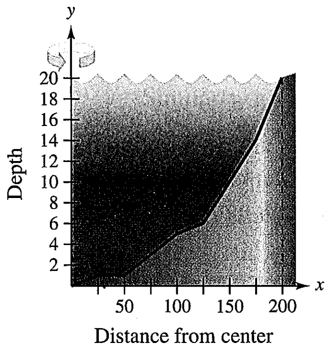
(b) Note that the roof line consists of two line segments. Find the equations of the line segments and use integration to find the volume of the shed.

58. Modeling Data A pond is approximately circular, with a diameter of 400 feet. Starting at the center, the depth of the water is measured every 25 feet and recorded in the table (see figure).

x	0	25	50
Depth	20	19	19

x	75	100	125
Depth	17	15	14

x	150	175	200
Depth	10	6	0



(a) Use Simpson's Rule to approximate the volume of water in the pond.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for the depths recorded in the table. Use the graphing utility to plot the depths and graph the model.

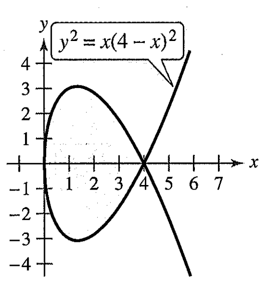
(c) Use the integration capabilities of a graphing utility and the model in part (b) to approximate the volume of water in the pond.

(d) Use the result of part (c) to approximate the number of gallons of water in the pond. (*Hint:* 1 cubic foot of water is approximately 7.48 gallons.)

59. Equal Volumes Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = 1/x$, $y = 0$, $x = \frac{1}{4}$, and $x = c$ (where $c > \frac{1}{4}$) is revolved about the x-axis and the y-axis, respectively. Find the value of c for which $V_1 = V_2$.

60. Volume of a Segment of a Paraboloid The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the y-axis to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring (a) by integrating with respect to x and (b) by integrating with respect to y.

61. Finding Volumes of Solids Consider the graph of $y^2 = x(4 - x)^2$ (see figure). Find the volumes of the solids that are generated when the loop of this graph is revolved about (a) the x-axis, (b) the y-axis, and (c) the line $x = 4$.



SECTION PROJECT

Saturn

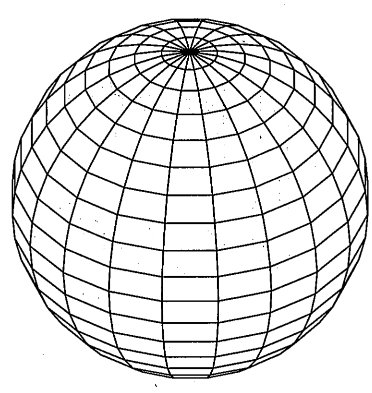
The Oblateness of Saturn Saturn is the most oblate of the planets in our solar system. Its equatorial radius is 60,268 kilometers and its polar radius is 54,364 kilometers. The color-enhanced photograph of Saturn was taken by Voyager 1. In the photograph, the oblateness of Saturn is clearly visible.

(a) Find the ratio of the volumes of the sphere and the oblate ellipsoid shown below.

(b) If a planet were spherical and had the same volume as Saturn, what would its radius be?

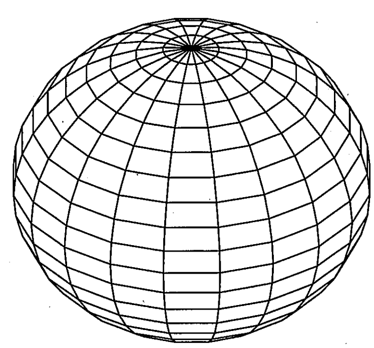
Computer model of "spherical Saturn," whose equatorial radius is equal to its polar radius. The equation of the cross section passing through the pole is

$$x^2 + y^2 = 60{,}268^2.$$



Computer model of "oblate Saturn," whose equatorial radius is greater than its polar radius. The equation of the cross section passing through the pole is

$$\frac{x^2}{60{,}268^2} + \frac{y^2}{54{,}364^2} = 1.$$



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