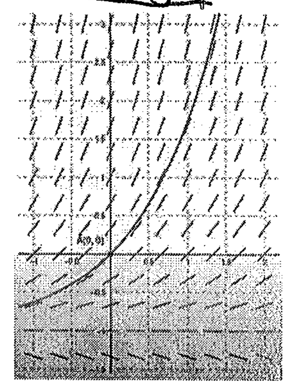
Key



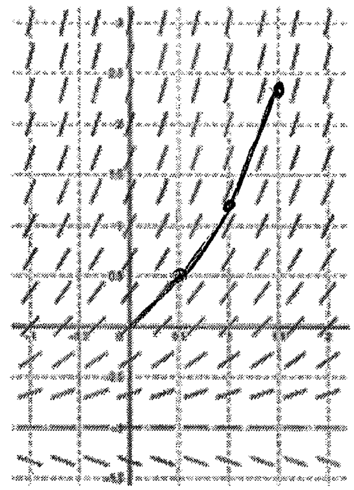
BC Calculus – 7.4 Notes – Euler's Method

In lesson 7.6, we will show how to find a solution $y = f(x)$ to a differential equation. In this lesson, we are going to APPROXIMATE a solution to a differential equation. This approximation method is called Euler's method.

this is the actual solution curve that is not given in the problem.

1. $\frac{dy}{dx} = 1 + y$ and $y(0) = 0$. $\Delta x = 0.5$. Using Euler's Method, show an approximation to the solution curve $y = f(x)$.

$\to$ step size



Step 1: Construct a tangent line at $(0, 0)$ for $0 \leq x \leq 0.5$.

$m = \frac{dy}{dx} = 1 + y$

$\left.\frac{dy}{dx}\right|_{(0,0)} = 1$

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = x$

Starting point was $(0, 0)$. New point to work with is $(0.5, 0.5)$

Step 2: Construct a tangent line at $(0.5, 0.5)$ **for $0.5 \leq x \leq 1$.**

$\left.\frac{dy}{dx}\right|_{(0.5,0.5)} = 1 + 0.5 = 1.5$

$y - 0.5 = m(x - 0.5)$

$y - 0.5 = 1.5(x - 0.5)$

$y - 0.5 = 1.5x - 0.75$

$y = 1.5x - 0.25$

$y(1) = 1.5(1) - 0.25$

$y(1) = 1.25$

Starting point was $(0.5, 0.5)$ **New point to work with is** $(1, 1.25)$

Step 3: Construct a tangent line at $(1, 1.25)$ **for $1 \leq x \leq 1.5$.**

$\left.\frac{dy}{dx}\right|_{(1,1.25)} = 2.25$

$y - 1.25 = 2.25(x - 1) = 2.25x - 2.25$

$y = 2.25x - 1$

$y(1.5) = 2.25(1.5) - 1 = 2.375$

Starting point was $(1, 1.25)$ **New point to work with is** $(1.5, 2.375)$

* The main idea with Euler's method is that by using multiple tangent lines (with updated slopes), then the approximation of the point on the solution curve will be closer to the actual value.
* The smaller the step size, the more accurate the approximation.

Here is a way Euler Method questions often appear on the AP Exam.

2. $\frac{dy}{dx} = 2x$ and let $f(x) = y$ be a solution to this differential equation. If $f(1) = 3$, what is the approximation to $f(2)$ obtained by using Euler's method with 5 steps of equal size?

First, find the step size. $\Delta x = \frac{b-a}{n} \rightarrow \Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$

$$y - y_1 = m(x - x_1)$$

$$y = y_1 + m(x - x_1)$$

* y' comes from differential equation dy/dx

new y = previous y + $y'(\Delta x)$ ← Δx is the step size

$\frac{dy}{dx} = 2x$

$\Delta x = 0.2$

x	y	y'	New y
1	3	2(1) = 2	3 + 2(0.2) = 3.4
1.2	3.4	2(1.2) = 2.4	3.4 + 2.4(0.2) = 3.88
1.4	3.88	2(1.4) = 2.8	3.88 + 2.8(0.2) = 4.44
1.6	4.44	2(1.6) = 3.2	4.44 + 3.2(0.2) = 5.08
1.8	5.08	2(1.8) = 3.6	5.08 + 3.6(0.2) = 5.8
2	$\boxed{5.8}$	~~ ~~	~~ ~~

$\boxed{f(2) \approx 5.8}$

Practice problems:

1. The table below gives the values of f', the derivative of f. If $f(1) = 2$, what is the approximation to $f(2.5)$ obtained by using Euler's method with 3 steps of equal size?

x	1	1.5	2.0	2.5
$f'(x)$	0.3	0.7	1.2	1.8

$\Delta x = 0.5$

x	y_0	y' or $f'(x)$	$y = y_0 + y'\Delta x$
1	2	0.3	$y = 2 + 0.3(0.5) = 2.15$
1.5	2.15	0.7	$y = 2.15 + 0.7(0.5) = 2.5$
2.0	2.5	1.2	$y = 2.5 + 1.2(0.5)$
2.5	$\boxed{3.1}$		y

$\boxed{f(2.5) \approx 3.1}$

2. The table below gives the values of f', the derivative of f. If $f(2) = 3$, what is the approximation to $f(2.6)$ obtained by using Euler's method with 2 steps of equal size?

x	2	2.3	2.6
$f'(x)$	−0.5	−0.3	−0.1

$\boxed{\Delta x = 0.3}$

x	y	y' (or f'(x))	y = y₀ + y'(Δx)
2	3	−0.5	$y = 3 + (-0.5)(0.3) = 2.85$
2.3	2.85	−0.3	$y = 2.85 + (-0.3)(0.3) = 2.76$
2.6	$\boxed{2.76}$		

$\boxed{f(2.6) \approx 2.76}$

3. The table below gives the values of f', the derivative of f. If $f(3) = 5$, what is the approximation to $f(4.0)$ obtained by using Euler's method with 2 steps of equal size?

x	3	3.25	3.5	3.75	4.0	4.25
$f'(x)$	0.1	0.3	0.5	0.7	0.9	1.1

$\boxed{\Delta x = 0.5}$

x	y_0	y' (or f'(x))	$y = y_0 + y'(\Delta x)$
3	5	0.1	$y = 5 + 0.1(0.5) = 5.05$
3.5	5.05	0.5	$y = 5.05 + 0.5(0.5) = 5.3$
4	$\boxed{5.3}$		

$\boxed{f(4) \approx 5.3}$

4. The table below gives the values of f', the derivative of f. If $f(1.5) = 4$, what is the approximation to $f(1)$ obtained by using Euler's method with 2 steps of equal size?

x	1	1.25	1.5	1.75	2.0
$f'(x)$	0.3	0.4	0.6	0.9	1.3

$\boxed{\Delta x = -0.25}$

x	y_0	y' (or f'(x))	$y = y_0 + y'(\Delta x)$
1.5	4	0.6	$y = 4 + 0.6(-0.25) = 3.85$
1.25	3.85	0.4	$y = 3.85 + 0.4(-0.25) = 3.75$
1	$\boxed{3.75}$		

$\boxed{f(1) \approx 3.75}$

5. Let $h(x) = \int_1^x \sqrt{1+t^2}\,dt$. Use Euler's method, starting at $x = 1$ with 2 steps of equal size, to approximate $h(3)$.

$\boxed{\Delta x = 1}$

X	h(x) or y_0	$h'(x) = \frac{d}{dx}\int_1^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$	$y = y_0 + y'\Delta x$
1	$\int_1^1 \sqrt{1+t^2} = h(1) = 0$	$h'(1) = \sqrt{1+1^2} = \sqrt{2}$	$y = 0 + \sqrt{2}(1) = \sqrt{2}$
2	$\sqrt{2}$	$h'(2) = \sqrt{1+2^2} = \sqrt{5}$	$y = \sqrt{2} + \sqrt{5}(1) = \sqrt{2}+\sqrt{5}$
3	$\boxed{\sqrt{2}+\sqrt{5}}$		

$\boxed{h(3) \approx \sqrt{2}+\sqrt{5}}$

$$h'(x) = \frac{d}{dx}\int_0^x \sqrt{1+3t^2}\,dt = \boxed{\sqrt{1+3x^2} = h'(x)}$$

6. Let $h(x) = \int_0^x \sqrt{1+3t^2}\,dt$. Use Euler's method, starting at $x = 0$ with 3 steps of equal size, to approximate $h(3)$.

$\boxed{\Delta x = 1}$

x	$h(x)$ or y_0	$h'(x)$ or y'	$y = y_0 + h'(x)(\Delta x)$
0	$h(0) = \int_0^0 \sqrt{1+3t^2} = 0$	$h'(0) = \sqrt{1+3(0)^2} = 1$	$y = 0 + 1(1) = 1$
1	1	$h'(1) = \sqrt{1+3(1)^2} = 2$	$y = 1 + 2(1) = 3$
2	3	$h'(2) = \sqrt{1+3(2)^2} = \sqrt{13}$	$y = 3 + \sqrt{13}(1) = 3+\sqrt{13}$
3	$\boxed{3+\sqrt{13}}$		

$\boxed{h(3) \approx 3+\sqrt{13}}$

7. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 2x - y$ with initial condition $f(1) = 0$. What is the approximation for $f(1.3)$ obtained using Euler's method with 3 steps of equal length, starting at $x = 1$?

$\Delta x = \frac{1.3-1}{3} = \frac{0.3}{3} = 0.1$

x	y_0	$y' = 2x - y$	$y = y_0 + y'(\Delta x)$
1	0	$y' = 2(1) - 0 = 2$	$y = 0 + 2(0.1) = 0.2$
1.1	0.2	$y' = 2(1.1) - 0.2 = 2$	$y = 0.2 + 2(0.1) = 0.4$
1.2	0.4	$y' = 2(1.2) - 0.4 = 2$	$y = 0.4 + 2(0.1) = 0.6$
1.3	$\boxed{0.6}$		

$\boxed{f(1.3) \approx 0.6}$

8. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ with initial condition $f(0) = 1$. What is the approximation for $f(.3)$ obtained using Euler's method with 3 steps of equal length, starting at $x = 0$?

$\boxed{\Delta x = 0.1}$

x	y_0	$y' = -x/y$	$y = y_0 + y'\Delta x$
0	1	$y' = \frac{-0}{1} = 0$	$1 + 0(0.1) = 1$
0.1	1	$y' = \frac{-0.1}{1} = -0.1$	$1 - 0.1(0.1) = 0.99$
0.2	0.99	$y' = -0.2/0.99 = -0.202$	$0.99 - 0.202(0.1) = 0.9698$
0.3	$\boxed{0.9698}$		

$\boxed{f(0.3) \approx 0.9698}$

9. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = y$ with initial condition $f(0) = 1$. What is the approximation for $f(.5)$ obtained using Euler's method with a step size of $\Delta x = 0.1$, starting at $x = 0$?

$\boxed{\Delta x = 0.1}$

x	y_0	$y' = y$	$y = y_0 + y'\Delta x$
0	1	1	$y = 1 + 1(0.1) = 1.1$
0.1	1.1	$y' = 1.1$	$y = 1.1 + 1.1(0.1) = 1.21$
0.2	1.21	$y' = 1.21$	$y = 1.21 + 1.21(0.1) = 1.331$
0.3	1.331	$y' = 1.331$	$y = 1.331 + 1.331(0.1) = 1.4641$
0.4	1.4641	$y' = 1.4641$	$y = 1.4641 + 1.4641(0.1) = 1.61051$
0.5	$\boxed{1.6105}$		

$\boxed{f(0.5) \approx 1.6105}$

10. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with initial condition $f(0) = 1$. What is the approximation for $f(.8)$ obtained using Euler's method with 4 steps of equal length, starting at $x = 0$?

x	y_0	$y' = x + y$	$y = y_0 + y'\Delta y$
0	1	$y' = 0 + 1$	$y = 1 + 1(0.2) = 1.2$
0.2	1.2	$y' = 0.2 + 1.2 = 1.4$	$y = 1.2 + 1.4(0.2) = 1.48$
0.4	1.48	$y' = 0.4 + 1.48 = 1.88$	$y = 1.48 + 1.88(0.2) = 1.856$
0.6	1.856	$y' = 0.6 + 1.856 = 2.456$	$y = 1.856 + 2.456(0.2) = 2.3472$
0.8	$\boxed{2.3472}$		

$\boxed{f(0.8) \approx 2.347}$