7.5 Homework – Logistic Models with Differential Equations

Pg. 565-566 #5, 9, 11, 15, 17, 19, 21, ~~25, 27~~, 29, 33

In Problems 5–10, for each logistic function P, identify

(a) the carrying capacity M.
(b) the maximum population growth rate k.
(c) the initial population $P_0 = P(0)$.
(d) the population P when P is growing most rapidly.

5. $P(t) = \dfrac{5500}{1 + 99e^{-0.02t}}$

5. To analyze the logistic function P, compare $P(t) = \frac{5500}{1+99e^{-0.02t}}$ to $P(t) = \frac{M}{1+ae^{-kt}}$

(a) $M = 5500$ is the carrying capacity.
(b) $k = 0.02$ is the maximum population growth rate.
(c) $P_0 = P(0)$ is found by evaluating the logistic function at $t = 0$.

$$P(0) = \frac{5500}{1 + 99e^{-0.02(0)}} = \frac{5500}{100} = 55$$

(d) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{5500}{2} = 2750$

9. $P(t) = \dfrac{150}{1 + 2e^{-0.2t}}$

To analyze the logistic function P, compare $P(t) = \frac{150}{1+2e^{-0.2t}}$ to $P(t) = \frac{M}{1+ae^{-kt}}$

(a) $M = 150$ is the carrying capacity.
(b) $k = 0.2$ is the maximum population growth rate.
(c) $P_0 = P(0)$ is found by evaluating the logistic function at $t = 0$.

$$P(0) = \frac{150}{1 + 2e^{-0.2(0)}} = \frac{150}{3} = 50$$

(d) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{150}{2} = 75$

In Problems 11–20, the given logistic differential equation models the rate of change of a population P with respect to time t. For each differential equation, find

(a) the carrying capacity M.
(b) the maximum population growth rate k.
(c) the population when P is growing most rapidly.

11. $\frac{dP}{dt} = 0.12P\left(1 - \frac{P}{1000}\right)$

To analyze the logistic differential equation $\frac{dP}{dt} = 0.12P\left(1 - \frac{P}{1000}\right)$, compare it to the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

(a) $M = 1000$ is the carrying capacity.
(b) $k = 0.12$ is the maximum population growth rate.
(c) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{1000}{2} = 500$

15. $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{2800}\right)$

To analyze the logistic differential equation $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{2800}\right)$, compare it to the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

(a) $M = 2800$ is the carrying capacity.
(b) $k = 0.002$ is the maximum population growth rate.
(c) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{2800}{2} = 1400$

17. $\dfrac{dP}{dt} = 0.00003P(1000 - P)$

To analyze the logistic differential equation $\frac{dP}{dt} = 0.00003P(1000 - P)$, compare it to the logistic differential equation $\frac{dP}{dt} = \frac{k}{M}P(M - P)$

(a) $M = 1000$ is the carrying capacity.

(b) $\frac{k}{1000} = 0.00003$ $k = 0.03$ is the maximum population growth rate.

(c) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{1000}{2} = 500$

19. $\dfrac{dP}{dt} = 0.0002P(1200 - P)$

To analyze the logistic differential equation $\frac{dP}{dt} = 0.0002P(1200 - P)$, compare it to the logistic differential equation $\frac{dP}{dt} = \frac{k}{M}P(M - P)$

(a) $M = 1200$ is the carrying capacity.

(b) $\frac{k}{1200} = 0.0002$ $k = 0.24$ is the maximum population growth rate.

(c) The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{1200}{2} = 600$

In Problems 21 and 22, write the logistic differential equation $\dfrac{dP}{dt}$ *whose solution is P.*

21. $P(t) = \dfrac{2100}{1 + 29e^{-0.15t}}$

To analyze the logistic equation $P(t) = \frac{2100}{1+29e^{-0.15t}}$, compare it to the logistic equation $P(t) = \frac{M}{1+ae^{-kt}}$ where $P(t)$ is the population at time t, M is the carrying capacity, k is the maximum population growth rate, $P_0 = P(0)$, and $a = \frac{M-P_0}{P_0}$.

Once we identify k and M we can substitute them into the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

For $P(t) = \frac{2100}{1+29e^{-0.15t}}$, $M = 2100$, $a = 29$, and $k = 0.15$.

So the logistic differential equation is $\boxed{\dfrac{dP}{dt} = 0.15P\left(1 - \dfrac{P}{2100}\right)}$

Omit #25 and #27

In Problems 29 and 30, for each logistic function find the population when it is growing most rapidly.

29. $P(t) = \dfrac{100}{1 + 2e^{-0.2t}}$

To analyze the logistic function P, compare $P(t) = \frac{100}{1+2e^{-0.02t}}$ to $P(t) = \frac{M}{1+ae^{-kt}}$
The Population, P, when P is growing most rapidly, is at the inflection point. This occurs when the population equals $\frac{M}{2} = \frac{100}{2} = \boxed{50}$

33. Spread of Flu Suppose one person with the flu is placed in a group of 49 people without the flu. The rate of change of those with the flu with respect to time t (in days) is proportional to the product of the number of those with the flu and the number without flu.

(a) If P is the number of people with the flu and the maximum daily population growth rate is 15%, write a differential equation that models the experiment.

(b) Write the logistic function that satisfies the differential equation.

(c) Find the time it takes for $\frac{1}{2}$ the population to become infected.

(d) How long does it take for 80% of the population to become infected?

(a) Since the rate of change of the number of people with the flu P with respect to time t (in days) is proportional to the product of P and $50 - P$, the differential equation can be written as $\frac{dP}{dt} = CP(50 - P)$. The differential equation can also be written in the form $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ where $M = 50$ is the carrying capacity and $k = 0.15$ is the maximum growth rate. The differential equation becomes $\boxed{\frac{dP}{dt} = 0.15P(1 - \frac{P}{50}) \text{ with } P(0) = 1}$.

(b) The initial number of people with the flu is $P_0 = 1$. The solution to the differential equation $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ is $P(t) = \frac{M}{1+ae^{-kt}}$ where $k = 0.15$, $M = 50$, and $a = \frac{M-P_0}{P_0} = \frac{50-1}{1} = 49$. Therefore, $\boxed{P(t) = \frac{50}{1+49e^{-0.15t}}}$.

(c) Find t so that $P(t) = 25$.

$$\frac{50}{1 + 49e^{-0.15t}} = 25$$

$$e^{-0.15t} = \frac{\frac{50}{25} - 1}{49}$$

$$-0.15t = \ln\left(\frac{\frac{50}{25} - 1}{49}\right)$$

$$t = -\frac{1}{0.15}\ln\left(\frac{1}{49}\right) = \frac{1}{0.15}\ln 49 \boxed{\approx 29.945 \text{ days}}.$$

33. Spread of Flu Suppose one person with the flu is placed in a group of 49 people without the flu. The rate of change of those with the flu with respect to time t (in days) is proportional to the product of the number of those with the flu and the number without flu.

(d) How long does it take for 80% of the population to become infected?

(d) Find t so that $P(t) = 0.80(50)$.

$$\begin{aligned}\frac{50}{1+49e^{-0.15t}} &= 40\\ e^{-0.15t} &= \frac{\frac{50}{40}-1}{49}\\ -0.15t &= \ln\left(\frac{0.25}{49}\right)\\ t &= -\frac{1}{0.15}\ln\left(\frac{0.25}{49}\right) \boxed{\approx 35.187 \text{ days}}\end{aligned}$$