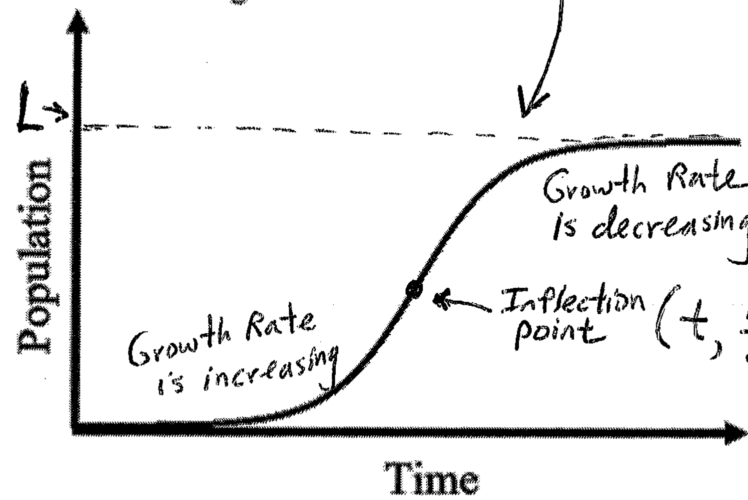
BC Calculus – 7.5 Notes – Logistic Models

Key

Logistic population growth is when the growth rate increases quickly at first, but then slows as the population reaches carrying capacity. (limiting value) → "L"

Graphs would look something like this:



$\left(t, \frac{L}{2}\right)$ → Max growth rate occurs at $y = \frac{L}{2}$

* Population is continually increasing but the growth rate is changing

A real-world application could be with animals and how many the land can support. (due to limits of natural resources)

Logistic Differential Equation

The derivative of a logistic function is typically written in one of the following forms:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

or if you manipulate this algebraically you could see it as

$$\frac{dy}{dt} = \frac{k}{L}y(L - y)$$

In either form, *k* and *L* are positive constants and *L* is the limiting value.

* dependent variable "y" shows up twice in the differential equation indicates the likelihood of logistic form.

Identify the limiting value and the *y*-value of the point of inflection for each solution of the given differential equation.

1. $\frac{dy}{dt} = 6y\left(1 - \frac{y}{3}\right)$ $1 - \frac{y}{L}$

   $\frac{L}{2}$

   Limiting value: 3

   ***y*-value of the pt of inflection:** $\frac{3}{2}$

2. $\frac{dy}{dt} = 2y(16 - y)$

   Limiting value: 16

   ***y*-value of the pt of inflection:** $\frac{L}{2} = \boxed{8}$

These two things will answer most questions you will see on the AP Exam for logistics.

1. **The maximum value of the logistic function is the limiting value.** → L value
2. **The maximum rate happens when y'' changes from positive to negative.**

   → at the inflection point $\left(\frac{L}{2}\right)$

Identify the carrying capacity and where the maximum rate of change occurs.

3. $\frac{dy}{dt} = 20y\left(1 - \frac{y}{100}\right)$ $\quad 1 - \frac{y}{L}$

Carrying capacity: 100

Maximum rate occurs at $y = 50$

4. $\frac{dy}{dt} = 4y(7 - y)$ $\quad L - y$

Carrying capacity: 7

Maximum rate occurs at $y = \frac{7}{2}$

You can derive a general solution, using separation of variables, to solve $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

You end up with something that looks like this: $y = \frac{L}{1+be^{-kt}}$ → Logistic Form of a standard function

5. The rate of change $\frac{dP}{dt}$ of the number of people entering a state park is modeled by a logistic differential equation. The capacity of the state park is 2500 people. At a certain time, the number of people in the state park is 1200 and is increasing at a rate of 100 people per hour. Create a differential equation that could represent this situation.

*use $\frac{dy}{dt} = \frac{k}{L} y[L-y]$

$\frac{dP}{dt} = \frac{K}{2500} \cdot P(2500-P)$

* when $P=1200$, $\frac{dP}{dt}=100$, find K

$100 = \frac{k}{2500} \cdot 1200(2500-1200)$

$100 = \frac{k}{2500} \cdot 1200 \cdot 1300$

$100 = \frac{12k}{25} \cdot 1300$

$\frac{100}{1300} = \frac{12}{25}k$

$\frac{1}{13} \cdot \frac{25}{12} = k$

$\frac{25}{156} = k$

$\frac{dP}{dt} = \frac{25}{156} \cdot \frac{1}{2500} \cdot P(2500-P)$

$\boxed{\frac{dP}{dt} = \frac{25}{156} P\left(1 - \frac{P}{2500}\right)}$

6. Find the carrying capacity $\frac{dy}{dt} = \frac{4}{5}y - \frac{1}{150}y^2$

* Find L:

put in form:

$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

$\frac{dy}{dt} = \frac{4}{5}y\left[1 - \frac{1}{120}y\right]$

$\frac{dy}{dt} = \frac{4}{5}y\left[1 - \frac{y}{120}\right]$

$\frac{1}{150} \div \frac{4}{5}$

$\frac{1}{150} \cdot \frac{5}{4} = \frac{1}{120}$

$\boxed{L = 120}$

Practice Problems:

1. A population y changes at a rate modeled by the logistic differential equation $\frac{dy}{dt} = 0.3y(4000 - y)$, where t is measured in years. What are all the values of y for which the population is increasing at a decreasing rate?

$y' > 0$ $\quad$ $y'' < 0$

$\frac{dy}{dt} = 1200y - 0.3y^2$

$\frac{d^2y}{dt^2} = 1200\left(\frac{dy}{dt}\right) - 0.6y\left(\frac{dy}{dt}\right)$

$0 = \frac{dy}{dt}(1200 - 0.6y)$

$0 = 1200 - 0.6y$

$0.6y = 1200$

$y = 2000$

y'': 0 (+) 2000 (−) 4000

y': 0 (+) 4000

$\boxed{2000 < y < 4000}$

0.7 is the capacity limit

2. A rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(0.7 - y)$, where y is the proportion of the population that has heard the rumor at time t.

   a. What proportion of the population has heard the rumor when it is spreading the fastest? → y' is a max, or y'' change signs

$$y' = 1.4y - 2y^2 \qquad 4y = 1.4$$
$$y'' = 1.4 - 4y \qquad \boxed{y = 0.35}$$
$$0 = 1.4 - 4y$$

   b. If at $t = 0$, 20% of the people have heard the rumor, find y as a function of t.

separation of variables

$$dy = 2y(0.7-y)\,dt$$
$$\frac{dy}{y(0.7-y)} = 2\,dt$$

partial fraction decomposition

$$\frac{A}{y} + \frac{B}{0.7-y} = \frac{1}{y(0.7-y)}$$
$$y=0 \quad ; \quad y=0.7$$
$$A = \frac{10}{7} ; \quad B = \frac{10}{7}$$
$$\int \frac{10}{7}\cdot\frac{1}{y}\,dy + \int \frac{10}{7}\cdot\frac{1}{0.7-y}\,dy = \int 2\,dt$$
$$u = 0.7 - y$$
$$\frac{du}{dy} = -1$$
$$\frac{10}{7}\ln|y| - \frac{10}{7}\ln|0.7-y| = 2t + C$$

$$\frac{10}{7}\ln\left|\frac{y}{0.7-y}\right| = 2t + C$$
$$e^{\ln\left|\frac{y}{0.7-y}\right|} = e^{\frac{14}{10}t + C}$$
$$\frac{y}{0.7-y} = e^{7/5\,t}\cdot e^{C}$$
$$\frac{y}{0.7-y} = Ce^{7/5\,t} \quad \leftarrow (t, y) \; (0, 0.2)$$
$$\frac{0.2}{0.7-0.2} = Ce^{0}$$
$$\frac{0.2}{0.5} = C$$
$$C = \frac{2}{5}$$

$$\frac{y}{0.7-y} = \frac{2}{5}e^{7/5\,t}$$
$$5y = 2e^{7/5\,t}(0.7-y)$$
$$5y = 1.4e^{7/5\,t} - 2ye^{7/5\,t}$$
$$5y + 2ye^{7/5\,t} = 1.4e^{7/5\,t}$$
$$y(5 + 2e^{7/5\,t}) = 1.4e^{7/5\,t}$$
$$\boxed{y = \frac{1.4e^{7/5\,t}}{5 + 2e^{7/5\,t}}}$$

   c. At what time t is the rumor spreading the fastest? [no calculator, give an exact answer.]

Max when $y = 0.35$

$$\frac{y}{0.7-y} = \frac{2}{5}e^{7/5\,t}$$
$$\frac{0.35}{0.7-0.35} = \frac{2}{5}e^{7/5\,t}$$
$$\frac{0.35}{0.35} = \frac{2}{5}e^{7/5\,t}$$

$$1 = \frac{2}{5}e^{7/5\,t}$$
$$\frac{5}{2} = e^{7/5\,t}$$
$$\ln(5/2) = \ln e^{7/5\,t}$$
$$\ln(5/2) = \frac{7}{5}t$$
$$\boxed{\frac{5}{7}\ln(5/2) = t}$$

3. The population P of a city at time t is increasing according to a logistic differential equation. Which of the following could be the differential equation?

↳ ex: $\frac{dy}{dt} = 2y(4-y)$

A. $\frac{dP}{dt} = 0.375t$

B. $\frac{dP}{dt} = 0.375t(15000 - t)$

C. $\frac{dP}{dt} = 0.375P$

D. $\frac{dP}{dt} = 0.375(15000 - P)$

E. $\boxed{\frac{dP}{dt} = 0.375P(15000 - P)}$

$\frac{dy}{dt} = \frac{k}{L}y(L-y)$

---

4. The total number of positive COVID cases in a city t days after the start of an outbreak is modeled by the function $y = C(t)$ that is the solution to the logistic differential equation $\frac{dy}{dt} = \frac{1}{7000}y(1600 - y)$. If there are 10 reported positive COVID cases initially, what is the limiting value for the total number of positive cases of the COVID virus as t increases?

↳ same as "capacity limit (L)

$\boxed{\text{1600 cases}}$

---

5. The size of a rabbit population is modeled by the function R that is a solution to the logistic differential equation $\frac{dR}{dt} = \frac{R}{3} - \frac{R^2}{2400}$, where t is measured in years for $t \geq 0$ and the initial population satisfies $R(0) > 0$. Which of the following statements could be true?

$\frac{dR}{dt} = \frac{1}{3}R\left[1 - \frac{R}{800}\right]$

$\star\ \frac{dy}{dt} = ky\left[1 - \frac{y}{L}\right]$

False
$L = 800$ → I. $\lim_{t\to\infty} R(t) > 1000$

✓ II. The graph of R has a point of inflection for $t > 0$.

✓ III. The maximum rate of change of R occurs at $t = 0$.

A. None

B. II only

C. I & II only

$\boxed{\text{D. II \& III only}}$

6. The rate of change $\frac{dP}{dt}$ of the number of people in a mall is modeled by a logistic differential equation. The maximum number of people allowed in the mall is 2000. At 10 A.M., the number of people in the mall is 200 and is increasing at a rate of 400 people per hour. Which of the following differential equations describe this situation?

A. $\frac{dP}{dt} = \frac{1}{400}(2000 - P) + 200$

B. $\frac{dP}{dt} = \frac{2}{5}(2000 - P)$

C. $\boxed{\frac{dP}{dt} = \frac{1}{900}P(2000 - P)}$

D. $\frac{dP}{dt} = 900P(2000 - P)$

E. $\frac{dP}{dt} = \frac{1}{400}P(2000 - P)$

* Limiting Value $L = 2000$

* when $P = 200$, $\frac{dP}{dt} = 400$

* $\frac{dP}{dt} = \frac{k}{L}P(L-P)$

$400 = \frac{k}{2000}(200)[2000-200] = \frac{1}{10}k(1800)$

$400 = 180k$

$\frac{20}{9} = k$

$\frac{dP}{dt} = \frac{20}{9} \cdot \frac{1}{2000} P(2000-P)$

---

7. The population P of deer in a preserve grows at a rate that is $\boxed{\text{jointly proportional}}$ to the size of the deer population and the difference between the deer population and the carrying capacity of the population. If the carrying capacity of the preserve is 3000 deer, which of the following differential equations best models the growth rate of the deer population with respect to time t, where k is a constant?

$L = 3000$

A. $\frac{dP}{dt} = 3000k(1 - P)$

B. $\frac{dP}{dt} = 3000 - kP$

C. $\frac{dP}{dt} = k(3000 - P)$

D. $\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{3000}\right)}$

E. $\frac{dP}{dt} = \frac{k}{P}(2000 - P)$

$\frac{dP}{dt} = S(L-P)$

$\downarrow$

$= \frac{k}{L}P(L-P)$

$\frac{dP}{dt} = \frac{k}{3000}P(3000-P)$

or

$\frac{dP}{dt} = kP\left(\frac{3000}{3000} - \frac{P}{3000}\right) = kP\left(1 - \frac{P}{3000}\right)$

---

8. The rate of change, $\frac{dP}{dt}$, of the number of people entering an arena is modeled by a logistic differential equation. The capacity of the arena is 5000 people. At a certain time, the number of people in the arena is 1000 and is increasing at the rate of 500 people per minute. Which of the following differential equations could describe this situation?

$L = 5000$ When $P = 1000$, $\frac{dP}{dt} = 500$

A. $\frac{dP}{dt} = \frac{1}{800}(5000 - P)$

B. $\frac{dP}{dt} = \frac{1}{500}P(5000 - P)$

C. $\boxed{\frac{dP}{dt} = \frac{1}{8000}P(5000 - P)}$

D. $\frac{dP}{dt} = \frac{1}{5000}P(500 - P)$

$\frac{dP}{dt} = \frac{k}{L}P(L-P)$

$500 = \frac{k}{5000}(1000)(5000-1000) \rightarrow \frac{k}{5}(4000) \rightarrow k(800)$

$500 = 800k$

$\frac{5}{8} = k$

$\frac{dP}{dt} = \frac{5}{8} \cdot \frac{1}{5000} P(5000-P)$

$\boxed{\frac{dP}{dt} = \frac{1}{800}P(5000-P)}$

9. If a certain population is modeled by the function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right)$, where t is the time in years and $P(0) = 100$. What is $\boxed{\lim_{t\to\infty} P(t)?}$ means limiting capacity

$\boxed{L=200}$

---

10. The function P satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{20}\left(1 - \frac{P}{1700}\right)$, where $P(0) = 210$. Which of the following statements is false?

✓A. $\lim_{t\to\infty} P(t) = 1700$ True, Limiting capacity $=1700$

$\boxed{\text{B. } \frac{dP}{dt} \text{ has a maximum value when } P = 210.}$ False, Max $\frac{dP}{dt}$ when $P=850$

✓C. $\frac{d^2P}{dt^2} = 0$ when $P = 850$ Max rate occurs at $\frac{L}{2} \rightarrow \frac{1700}{2} = 850$ ✓

✓D. When $P > 850, \frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

positive slope

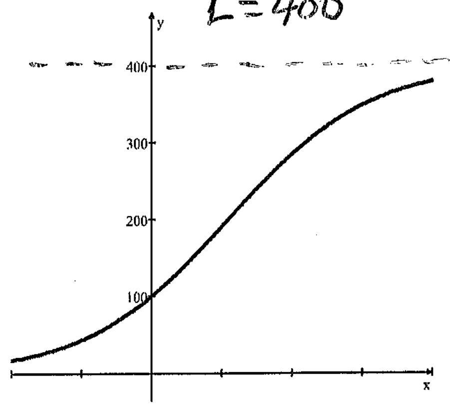
concave down

$(\_, 850)$

---

11. Which of the following differential equations for a population P could model the logistic growth shown in the figure?

L=400



$\boxed{\text{A. } \frac{dP}{dt} = 0.1P - 0.00025P^2}$

B. $\frac{dP}{dt} = 0.1P - 0.025P^2$

C. $\frac{dP}{dt} = 0.1P^2 - 0.00025P$

D. $\frac{dP}{dt} = 0.1P + 0.00025P^2$

E. $\frac{dP}{dt} = 0.1P + 0.025P^2$

$\frac{dP}{dt} = \frac{k}{L}P[L-P]$

$= \frac{k}{400}P(400-P)$

$= kP - \frac{kP^2}{400}$

$k=0.1$

$=0.1P - \frac{0.1P^2}{400}$

$= \boxed{0.1P - 0.00025P^2}$