97. On $\left[0, \frac{\pi}{2}\right]$, $\sin x \le 1 \Rightarrow x \sin x \le x \Rightarrow \int_0^{\pi/2} x \sin x\, dx \le \int_0^{\pi/2} x\, dx$.

98. (a) $A = \int_0^{\pi} x \sin x\, dx = \left[\sin x - x \cos x\right]_0^{\pi} = \pi$

(b) $\int_{\pi}^{2\pi} x \sin x\, dx = \left[\sin x - x \cos x\right]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

$A = 3\pi$

(c) $\int_{2\pi}^{3\pi} x \sin x\, dx = \left[\sin x - x \cos x\right]_{2\pi}^{3\pi} = 3\pi + 2\pi = 5\pi$

$A = 5\pi$

The area between $y = x \sin x$ and $y = 0$ on $\left[n\pi, (n+1)\pi\right]$ is $(2n+1)\pi$:

$\int_{n\pi}^{(n+1)\pi} x \sin x\, dx = \left[\sin x - x \cos x\right]_{n\pi}^{(n+1)\pi} = \pm(n+1)\pi \pm n\pi = \pm(2n+1)\pi$

$A = \left|\pm(2n+1)\pi\right| = (2n+1)\pi$

99. For any integrable function, $\int f(x)dx = C + \int f(x)dx$, but this cannot be used to imply that $C = 0$.

Section 8.3 Trigonometric Integrals

1. Let $u = \cos x$, $du = -\sin x\, dx$.

$$\int \cos^5 x \sin x\, dx = -\int \cos^5 x(-\sin x)\, dx = -\frac{\cos^6 x}{6} + C$$

2. $\int \cos^3 x \sin^4 x\, dx = \int \cos x\left(1 - \sin^2 x\right)\sin^4 x\, dx = \int\left(\sin^4 x - \sin^6 x\right)\cos x\, dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

3. Let $u = \sin 2x$, $du = 2\cos 2x\, dx$.

$$\int \sin^7 2x \cos 2x\, dx = \frac{1}{2}\int \sin^7 2x(2\cos 2x)\, dx = \frac{1}{2}\left(\frac{\sin^8 2x}{8}\right) + C = \frac{1}{16}\sin^8 2x + C$$

4.
$$\int \sin^3 3x\, dx = \int \sin^2 3x \sin 3x\, dx = \int\left(1 - \cos^2 3x\right)\sin 3x\, dx = \int \sin 3x\, dx - \int \cos^2 3x(\sin 3x\, dx) = -\frac{1}{3}\cos 3x + \frac{\cos^3 3x}{9} + C$$

5.
$$\int \sin^3 x \cos^2 x\, dx = \int\left(1 - \cos^2 x\right)\cos^2 x \sin x\, dx = \int\left(\cos^2 x - \cos^4 x\right)\sin x\, dx = -\int\left(\cos^2 x - \cos^4 x\right)(-\sin x)\, dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

6. Let $u = \sin\frac{x}{3}$, $du = \frac{1}{3}\cos\frac{x}{3}\, dx$.

$$\int \cos^3\frac{x}{3}\, dx = \int\left(\cos\frac{x}{3}\right)\left(1 - \sin^2\frac{x}{3}\right) dx = 3\int\left(1 - \sin^2\frac{x}{3}\right)\left(\frac{1}{3}\cos\frac{x}{3}\right) dx = 3\left(\sin\frac{x}{3} - \frac{1}{3}\sin^3\frac{x}{3}\right) + C = 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

7. $\int \sin^3 2\theta\sqrt{\cos 2\theta}\,d\theta = \int(1-\cos^2 2\theta)\sqrt{\cos 2\theta}\sin 2\theta\,d\theta$

$$= \int\left[(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}\right]\sin 2\theta\,d\theta$$

$$= -\frac{1}{2}\int\left[(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}\right](-2\sin 2\theta)\,d\theta$$

$$= -\frac{1}{2}\left[\frac{2}{3}(\cos 2\theta)^{3/2} - \frac{2}{7}(\cos 2\theta)^{7/2}\right] + C$$

$$= -\frac{1}{3}(\cos 2\theta)^{3/2} + \frac{1}{7}(\cos 2\theta)^{7/2} + C$$

8. $\int \frac{\cos^5 t}{\sqrt{\sin t}}\,dt = \int \cos t(1-\sin^2 t)^2(\sin t)^{-1/2}\,dt$

$$= \int(1 - 2\sin^2 t + \sin^4 t)(\sin t)^{-1/2}\cos t\,dt$$

$$= \int\left[(\sin t)^{-1/2} - 2(\sin t)^{3/2} + (\sin t)^{7/2}\right]\cos t\,dt$$

$$= 2\sqrt{\sin t} - \frac{4}{5}(\sin t)^{5/2} + \frac{2}{9}(\sin t)^{9/2} + C$$

9. $\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C = \frac{1}{12}(6x + \sin 6x) + C$

10. $\int \sin^4 6\theta\,d\theta = \int\left(\frac{1-\cos 12\theta}{2}\right)\left(\frac{1-\cos 12\theta}{2}\right)d\theta$

$$= \frac{1}{4}\int(1 - 2\cos 12\theta + \cos^2 12\theta)\,d\theta$$

$$= \frac{1}{4}\int\left(1 - 2\cos 12\theta + \frac{1+\cos 24\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\int\left(\frac{3}{2} - 2\cos 12\theta + \frac{1}{2}\cos 24\theta\right)d\theta$$

$$= \frac{1}{4}\left(\frac{3}{2}\theta - \frac{1}{6}\sin 12\theta + \frac{1}{48}\sin 24\theta\right) + C = \frac{3}{8}\theta - \frac{1}{24}\sin 12\theta + \frac{1}{192}\sin 24\theta + C$$

11. Integration by parts:

$dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$

$u = x \Rightarrow du = dx$

$\int x\sin^2 x\,dx = \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\int(2x - \sin 2x)\,dx$

$$= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\left(x^2 + \frac{1}{2}\cos 2x\right) + C = \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$$

12. Use integration by parts twice.

$$dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x^2 \Rightarrow du = 2x\,dx$$

$$dv = \sin 2x\,dx \quad\Rightarrow\quad v = -\frac{1}{2}\cos 2x$$

$$u = x \quad\Rightarrow\quad du = dx$$

$$\begin{aligned}\int x^2 \sin^2 x\,dx &= \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int(2x^2 - x\sin 2x)\,dx \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x\sin 2x\,dx \\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int\cos 2x\,dx\right) \\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C \\ &= \frac{1}{24}(4x^3 - 6x^2\sin 2x - 6x\cos 2x + 3\sin 2x) + C\end{aligned}$$

13. $\int_0^{\pi/2}\cos^7 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \frac{16}{35}, (n = 7)$

14. $\int_0^{\pi/2}\cos^9 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{128}{315}, (n = 9)$

15. $\int_0^{\pi/2}\cos^{10} x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right)$

$= \frac{63}{512}\pi, (n = 10)$

16. $\int_0^{\pi/2}\sin^5 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}, \quad (n = 5)$

17. $\int_0^{\pi/2}\sin^6 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\frac{\pi}{2} = \frac{5\pi}{32}, (n = 6)$

18. $\int_0^{\pi/2}\sin^8 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right) = \frac{35\pi}{256}, (n = 8)$

19. $\int\sec 4x\,dx = \frac{1}{4}\int\sec 4x(4\,dx)$

$= \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$

20. $\int\sec^4 2x\,dx = \int(1 + \tan^2 2x)\sec^2 2x\,dx$

$= \frac{1}{2}\tan 2x + \frac{\tan^3 2x}{6} + C$

21. $dv = \sec^2 \pi x\,dx \quad\Rightarrow\quad v = \frac{1}{\pi}\tan \pi x$

$u = \sec \pi x \quad\Rightarrow\quad du = \pi\sec\pi x\tan\pi x\,dx$

$$\int\sec^3 \pi x\,dx = \frac{1}{\pi}\sec\pi x\tan\pi x - \int\sec\pi x\tan^2\pi x\,dx = \frac{1}{\pi}\sec\pi x\tan\pi x - \int\sec\pi x(\sec^2\pi x - 1)\,dx$$

$$2\int\sec^3\pi x\,dx = \frac{1}{\pi}(\sec\pi x\tan\pi x + \ln|\sec\pi x + \tan\pi x|) + C_1$$

$$\int\sec^3\pi x\,dx = \frac{1}{2\pi}(\sec\pi x\tan\pi x + \ln|\sec\pi x + \tan\pi x|) + C$$

22.
$$\begin{aligned}\int\tan^6 3x\,dx &= \int(\sec^2 3x - 1)\tan^4 3x\,dx \\ &= \int\tan^4 3x\sec^2 3x\,dx - \int\tan^4 3x\,dx \\ &= \int\tan^4 3x\sec^2 3x\,dx - \int\tan^2 3x(\sec^2 3x - 1)\,dx \\ &= \int\tan^4 3x\sec^2 3x\,dx - \int\tan^2 3x\sec^2 3x\,dx + \int(\sec^2 3x + 1)\,dx \\ &= \frac{\tan^5 3x}{15} - \frac{\tan^3 3x}{9} + \frac{\tan 3x}{3} + x + C\end{aligned}$$

23. $\int \tan^5 \frac{x}{2}\,dx = \int\left(\sec^2 \frac{x}{2} - 1\right)\tan^3 \frac{x}{2}\,dx$

$= \int \tan^3 \frac{x}{2}\sec^2 \frac{x}{2}\,dx - \int \tan^3 \frac{x}{2}\,dx$

$= \frac{\tan^4 \frac{x}{2}}{2} - \int\left(\sec^2 \frac{x}{2} - 1\right)\tan \frac{x}{2}\,dx$

$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2\ln\left|\cos \frac{x}{2}\right| + C$

24. $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$

25. Let $u = \sec 2t$, $du = 2\sec 2t \tan 2t$.

$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int\left(\sec^2 2t - 1\right)\sec^3 2t \cdot \tan 2t\,dt$

$= \int\left(\sec^4 2t - \sec^2 2t\right)\left(\sec 2t \tan 2t\right)dt = \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$

26. $\int \tan^5 2x \sec^4 2x\,dx = \int \tan^5 2x\left(\tan^2 2x + 1\right)\sec^2 2x\,dx$

$= \int \tan^7 2x \sec^2 2x\,dx + \int \tan^5 2x \sec^2 2x\,dx$

$= \frac{1}{2}\left(\frac{\tan^8 2x}{8}\right) + \frac{1}{2}\left(\frac{\tan^6 2x}{6}\right) + C$

$= \frac{\tan^8 2x}{16} + \frac{\tan^6 2x}{12} + C$

27. $\int \sec^6 4x \tan 4x\,dx = \frac{1}{4}\int \sec^5 4x\left(4\sec 4x \tan 4x\right)dx$

$= \frac{\sec^6 4x}{24} + C$

28. $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\sec \frac{x}{2}\tan \frac{x}{2}\right)dx$

$= \sec^2 \frac{x}{2} + C$ or

$\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \tan \frac{x}{2}\left(\frac{1}{2}\sec^2 \frac{x}{2}\right)dx = \tan^2 \frac{x}{2} + C$

29. $\int \sec^5 x \tan^3 x\,dx = \int \sec^4 x \tan^2 x\left(\sec x \tan x\right)dx$

$= \int \sec^4 x\left(\sec^2 x - 1\right)\left(\sec x \tan x\right)dx$

$= \int\left(\sec^6 x - \sec^4 x\right)\left(\sec x \tan x\right)dx$

$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

30. $\int \tan^3 3x\,dx = \int\left(\sec^2 3x - 1\right)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x\left(3\sec^2 3x\right)dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln\left|\cos 3x\right| + C$

31. $\int \frac{\tan^2 x}{\sec x}\,dx = \int \frac{\left(\sec^2 x - 1\right)}{\sec x}\,dx$

$= \int\left(\sec x - \cos x\right)dx$

$= \ln\left|\sec x + \tan x\right| - \sin x + C$

32. $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x\left(1 - \sin^2 x\right)\cos x\,dx$

$= \int\left(\sin^2 x - \sin^4 x\right)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

33. $r = \int \sin^4(\pi\theta)\,d\theta = \frac{1}{4}\int\left[1 - \cos(2\pi\theta)\right]^2 d\theta$

$= \frac{1}{4}\int\left[1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)\right]d\theta$

$= \frac{1}{4}\int\left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2}\right]d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{\pi}\sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi}\sin(4\pi\theta)\right] + C$

$= \frac{1}{32\pi}\left[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)\right] + C$

34. $s = \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}\, d\alpha$

$= \int \left(\frac{1 - \cos \alpha}{2}\right)\left(\frac{1 + \cos \alpha}{2}\right) d\alpha = \int \frac{1 - \cos^2 \alpha}{4}\, d\alpha$

$= \frac{1}{4}\int \sin^2 \alpha\, d\alpha = \frac{1}{8}\int (1 - \cos 2\alpha)\, d\alpha$

$= \frac{1}{8}\left(\theta - \frac{\sin 2\alpha}{2}\right) + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

35. $y = \int \tan^3 3x \sec 3x\, dx$

$= \int (\sec^2 3x - 1) \sec 3x \tan 3x\, dx$

$= \frac{1}{3}\int \sec^2 3x(3 \sec 3x \tan 3x)\, dx - \frac{1}{3}\int 3 \sec 3x \tan 3x\, dx$

$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

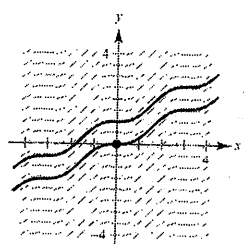
36. $y = \int \sqrt{\tan x} \sec^4 x\, dx$

$= \int \tan^{1/2} x(\tan^2 x + 1) \sec^2 x\, dx$

$= \int (\tan^{5/2} x + \tan^{1/2} x) \sec^2 x\, dx$

$= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C$

37. (a)

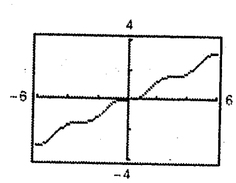


(b) $\frac{dy}{dx} = \sin^2 x, \quad (0, 0)$

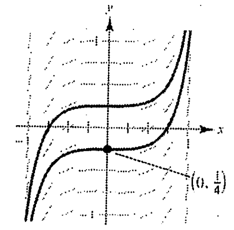
$y = \int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$

$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$

$(0, 0)$: $0 = C, y = \frac{1}{2}x - \frac{\sin 2x}{4}$



38. (a)

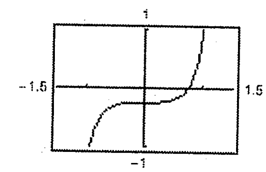


(b) $\frac{dy}{dx} = \sec^2 x \tan^2 x, \quad \left(0, -\frac{1}{4}\right)$

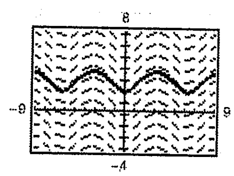
$y = \int \sec^2 x \tan^2 x\, dx \qquad u = \tan x, du = \sec^2 x\, dx$

$y = \frac{\tan^3 x}{3} + C$

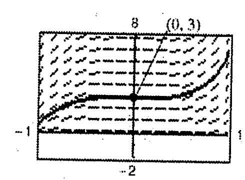
$\left(0, -\frac{1}{4}\right)$: $-\frac{1}{4} = C \Rightarrow y = \frac{1}{3}\tan^3 x - \frac{1}{4}$



39. $\frac{dy}{dx} = \frac{3 \sin x}{y}, y(0) = 2$



40. $\frac{dy}{dx} = 3\sqrt{y} \tan^2 x, y(0) = 3$



41. $\int \cos 2x \cos 6x\,dx = \frac{1}{2}\int\left[\cos\left((2-6)x\right) + \cos\left((2+6)x\right)\right]dx$

$= \frac{1}{2}\int\left[\cos(-4x) + \cos 8x\right]dx$

$= \frac{1}{2}\int(\cos 4x + \cos 8x)\,dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 8x}{8}\right] + C$

$= \frac{\sin 4x}{8} + \frac{\sin 8x}{16} + C$

$= \frac{1}{16}(2\sin 4x + \sin 8x) + C$

42. $\int \cos(5\theta)\cos(3\theta)\,d\theta = \frac{1}{2}\int\left[\cos(5-3)\theta + \cos(5+3)\theta\right]d\theta$

$= \frac{1}{2}\int(\cos 2\theta + \cos 8\theta)\,d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \frac{\sin 8\theta}{8}\right] + C$

$= \frac{\sin 2\theta}{4} + \frac{\sin 8\theta}{16} + C$

43. $\int \sin 2x \cos 4x\,dx = \frac{1}{2}\int\left[\sin\left((2-4)x\right) + \sin\left((2+4)x\right)\right]dx$

$= \frac{1}{2}\int\left(\sin(-2x) + \sin 6x\right)dx$

$= \frac{1}{2}\int(-\sin 2x + \sin 6x)\,dx$

$= \frac{1}{2}\left[\frac{\cos 2x}{2} - \frac{\cos 6x}{6}\right] + C$

$= \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

$= \frac{1}{12}(3\cos 2x - \cos 6x) + C$

44. $\int \sin(-7x)\cos(6x)\,dx = -\int \sin 7x \cos 6x\,dx$

$= -\frac{1}{2}\int\left[\sin(7-6)x + \sin(7+6)x\right]dx$

$= -\frac{1}{2}\int(\sin x + \sin 13x)\,dx$

$= -\frac{1}{2}\left[-\cos x - \frac{\cos 13x}{13}\right] + C$

$= \frac{1}{2}\cos x + \frac{1}{26}\cos 13x + C$

45. $\int \sin\theta \sin 3\theta\,d\theta = \frac{1}{2}\int(\cos 2\theta - \cos 4\theta)\,d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) + C$

$= \frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

46. $\int \sin 5x \sin 4x\,dx = \frac{1}{2}\int(\cos x - \cos 9x)\,dx$

$$= \frac{1}{2}\left(\sin x - \frac{\sin 9x}{9}\right) + C$$
$$= \frac{\sin x}{2} - \frac{\sin 9x}{18} + C$$
$$= \frac{1}{18}(9 \sin x - \sin 9x) + C$$

47. $\int \cot^3 2x\,dx = \int(\csc^2 2x - 1)\cot 2x\,dx$

$$= -\frac{1}{2}\int \cot 2x(-2\csc^2 2x)\,dx - \frac{1}{2}\int\frac{2\cos 2x}{\sin 2x}\,dx$$
$$= -\frac{1}{4}\cot^2 2x - \frac{1}{2}\ln|\sin 2x| + C$$
$$= \frac{1}{4}\left(\ln|\csc^2 2x| - \cot^2 2x\right) + C$$

48. $\int \tan^5 \frac{x}{4}\sec^4 \frac{x}{4}\,dx$

$$= \int \tan^5 \frac{x}{4}\left(\tan^2 \frac{x}{4} + 1\right)\sec^2 \frac{x}{4}\,dx$$
$$= \int\left(\tan^7 \frac{x}{4} + \tan^5 \frac{x}{4}\right)\sec^2 \frac{x}{4}\,dx$$
$$= \frac{\tan^8 \frac{x}{4}}{2} + \frac{2\tan^6 \frac{x}{4}}{3} + C$$
$$= \frac{1}{2}\tan^8 \frac{x}{4} + \frac{2}{3}\tan^6 \frac{x}{4} + C$$

49. $\int \csc^4 3x\,dx = \int \csc^2 3x(1 + \cot^2 3x)\,dx$

$$= \int \csc^2 3x\,dx + \int \cot^2 3x \csc^2 3x\,dx$$
$$= -\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + C$$

50. $\int \cot^3 \frac{x}{2}\csc^4 \frac{x}{2}\,dx = \int \cot^2 \frac{x}{2}\csc^3 \frac{x}{2}\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$

$$= \int\left(\csc^2 \frac{x}{2} - 1\right)\csc^3 \frac{x}{2}\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$$
$$= \int\left(\csc^5 \frac{x}{2} - \csc^3 \frac{x}{2}\right)\left(\csc \frac{x}{2}\cot \frac{x}{2}\right)dx$$
$$= -\frac{1}{3}\csc^6 \frac{x}{2} + \frac{1}{2}\csc^4 \frac{x}{2} + C$$

51. $\int\frac{\cot^2 t}{\csc t}\,dt = \int\frac{\csc^2 t - 1}{\csc t}\,dt$

$$= \int(\csc t - \sin t)\,dt$$
$$= \ln|\csc t - \cot t| + \cos t + C$$

52. $\int\frac{\cot^3 t}{\csc t}\,dt = \int\frac{\cos^3 t}{\sin^2 t}\,dt = \int\frac{(1 - \sin^2 t)\cos t}{\sin^2 t}\,dt$

$$= \int\frac{\cos t}{\sin^2 t}\,dt - \int \cos t\,dt$$
$$= \frac{-1}{\sin t} - \sin t + C = -\csc t - \sin t + C$$

53. $\int\frac{1}{\sec x \tan x}\,dx = \int\frac{\cos^2 x}{\sin x}\,dx = \int\frac{1 - \sin^2 x}{\sin x}\,dx$

$$= \int(\csc x - \sin x)\,dx$$
$$= \ln|\csc x - \cot x| + \cos x + C$$

54. $\int\frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \int\frac{1 - 2\cos^2 x}{\cos x}\,dx$

$$= \int(\sec x - 2\cos x)\,dx$$
$$= \ln|\sec x + \tan x| - 2\sin x + C$$

55. $\int(\tan^4 t - \sec^4 t)\,dt = \int(\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t)\,dt, \qquad (\tan^2 t - \sec^2 t = -1)$

$$= -\int(\tan^2 t + \sec^2 t)\,dt = -\int(2\sec^2 t - 1)\,dt = -2\tan t + t + C$$

56. $\int\frac{1-\sec t}{\cos t - 1}\,dt = \int\frac{\cos t - 1}{(\cos t - 1)\cos t}\,dt = \int\sec t\,dt = \ln|\sec t + \tan t| + C$

57. $\int_{-\pi}^{\pi}\sin^2 x\,dx = 2\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$$

58. $\int_0^{\pi/3}\tan^2 x\,dx = \int_0^{\pi/3}(\sec^2 x - 1)\,dx$

$$= \left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$$

59. $\int_0^{\pi/4} 6\tan^3 x\,dx = 6\int_0^{\pi/4}(\sec^2 x - 1)\tan x\,dx$

$$= 6\int_0^{\pi/4}\left[\tan x\sec^2 x - \tan x\right]dx$$

$$= 6\left[\frac{\tan^2 x}{2} + \ln|\cos x|\right]_0^{\pi/4}$$

$$= 6\left[\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right)\right] = 6\left(\frac{1}{2} - \ln\sqrt{2}\right)$$

$$= 3(1 - \ln 2)$$

60. $\int_0^{\pi/3}\sec^{3/2} x\tan x\,dx = \int_0^{\pi/3}\sec^{1/2} x\,(\sec x\tan x)\,dx$

$$= \left[\frac{2}{3}\sec^{3/2} x\right]_0^{\pi/3}$$

$$= \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

61. Let $u = 1 + \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi/2}\frac{\cos t}{1+\sin t}\,dt = \left[\ln|1+\sin t|\right]_0^{\pi/2} = \ln 2$$

62. $\int_{\pi/6}^{\pi/3}\sin 6x\cos 4x\,dx = \frac{1}{2}\int_{\pi/6}^{\pi/3}(\sin 2x + \sin 10x)\,dx$

$$= \left[-\frac{\cos 2x}{4} - \frac{\cos 10x}{20}\right]_{\pi/6}^{\pi/3}$$

$$= \left(\frac{1}{8} + \frac{1}{40}\right) - \left(-\frac{1}{8} - \frac{1}{40}\right) = \frac{3}{10}$$

63. $\int_{-\pi/2}^{\pi/2} 3\cos^3 x\,dx = 3\int_{-\pi/2}^{\pi/2}(1 - \sin^2 x)\cos x\,dx$

$$= 3\left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2}$$

$$= 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = 4$$

64. $\int_{-\pi/2}^{\pi/2}(\sin^2 x + 1)\,dx = \int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 2x}{2} + 1\right)dx$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2} - \frac{1}{2}\cos 2x\right)dx$$

$$= \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

65. (a) Save one sine factor and convert the remaining factors to cosines. Then expand and integrate.

(b) Save one cosine factor and convert the remaining factors to sines. Then expand and integrate.

(c) Make repeated use of the power reducing formulas to convert the integrand to odd powers of the cosine. Then proceed as in part (b).

90. $f(x) = \sum_{i=1}^{N} a_i \sin(ix)$

(a)
$$f(x)\sin(nx) = \left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi}\left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} a_n \sin^2(nx)\,dx \quad \text{(by Exercise 89)}$$

$$= \int_{-\pi}^{\pi} a_n \frac{1-\cos(2nx)}{2}\,dx = \left[\frac{a_n}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\right]_{-\pi}^{\pi} = \frac{a_n}{2}(\pi + \pi) = a_n\pi$$

So, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$

(b) $f(x) = x$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x\,dx = 2$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin 2x\,dx = -1$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin 3x\,dx = \frac{2}{3}$$

Section 8.4 Trigonometric Substitution

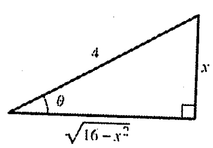
1. Use $x = 3\tan\theta$.

2. Use $x = 2\sin\theta$.

3. Use $x = 5\sin\theta$.

4. Use $x = 5\sec\theta$.

5. Let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$, $\sqrt{16-x^2} = 4\cos\theta$.



$$\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{4\cos\theta}{(4\cos\theta)^3}\,d\theta = \frac{1}{16}\int \sec^2\theta\,d\theta = \frac{1}{16}\tan\theta + C = \frac{1}{16}\left(\frac{x}{\sqrt{16-x^2}}\right) + C$$

6. Same substitution as in Exercise 5

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = 4\int \frac{4\cos\theta}{(4\sin\theta)^2(4\cos\theta)}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{16-x^2}}{x} + C = \frac{-\sqrt{16-x^2}}{4x} + C$$

7. Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{\sqrt{16 - x^2}}{x}\,dx &= \int \frac{4\cos\theta}{4\sin\theta}\,4\cos\theta\,d\theta \\
&= 4\int \frac{\cos^2\theta}{\sin\theta}\,d\theta \\
&= 4\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta \\
&= 4\int (\csc\theta - \sin\theta)\,d\theta \\
&= -4\ln\left|\csc\theta + \cot\theta\right| + 4\cos\theta + C \\
&= -4\ln\left|\frac{4}{x} + \frac{\sqrt{16 - x^2}}{x}\right| + 4\frac{\sqrt{16 - x^2}}{4} + C \\
&= -4\ln\left|\frac{4 + \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C \\
&= 4\ln\left|\frac{4 - \sqrt{16 - x^2}}{x}\right| + \sqrt{16 - x^2} + C
\end{aligned}$$

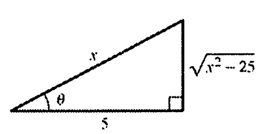
8. Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{x^3}{\sqrt{16 - x^2}}\,dx &= \int \frac{(4\sin\theta)^3}{4\cos\theta}\,4\cos\theta\,d\theta \\
&= 64\int \sin^3\theta\,d\theta \\
&= 64\int \left(1 - \cos^2\theta\right)\sin\theta\,d\theta \\
&= 64\left[-\cos\theta + \frac{\cos^3\theta}{3}\right] + C \\
&= 64\left[-\frac{\sqrt{16 - x^2}}{4} + \frac{\left(16 - x^2\right)^{3/2}}{64(3)}\right] + C \\
&= -16\sqrt{16 - x^2} + \frac{1}{3}\left(16 - x^2\right)^{3/2} + C \\
&= -\frac{1}{3}\sqrt{16 - x^2}\left[48 - \left(16 - x^2\right)\right] + C \\
&= -\frac{1}{3}\sqrt{16 - x^2}\left(32 + x^2\right) + C
\end{aligned}$$

9. Let $x = 5\sec\theta, dx = 5\sec\theta\tan\theta\,d\theta,$

$\sqrt{x^2 - 25} = 5\tan\theta.$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - 25}}\,dx &= \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln\left|\sec\theta + \tan\theta\right| + C \\
&= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5}\right| + C \\
&= \ln\left|x + \sqrt{x^2 - 25}\right| + C
\end{aligned}$$



10. Same substitution as in Exercise 9

$$\int \frac{\sqrt{x^2 - 25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta} 5\sec\theta\tan\theta\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta$$
$$= 5\int (\sec^2\theta - 1)\,d\theta$$
$$= 5(\tan\theta - \theta) + C$$
$$= 5\left(\frac{\sqrt{x^2 - 25}}{5} - \operatorname{arcsec}\frac{x}{5}\right) + C$$
$$= \sqrt{x^2 - 25} - 5\operatorname{arcsec}\frac{x}{5} + C$$

$$\left[\textbf{Note: } \operatorname{arcsec}\left(\frac{x}{5}\right) = \arctan\left(\frac{\sqrt{x^2 - 25}}{5}\right)\right]$$

11. Same substitution as in Exercise 9

$$\int x^3\sqrt{x^2 - 25}\,dx = \int (5\sec\theta)^3(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta$$
$$= 3125\int \sec^4\theta\tan^2\theta\,d\theta$$
$$= 3125\int (1 + \tan^2\theta)\tan^2\theta\sec^2\theta\,d\theta$$
$$= 3125\int (\tan^2\theta + \tan^4\theta)\sec^2\theta\,d\theta$$
$$= 3125\left[\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right] + C$$
$$= 3125\left[\frac{(x^2 - 25)^{3/2}}{125(3)} + \frac{(x^2 - 25)^{5/2}}{5^5(5)}\right] + C$$
$$= \frac{1}{15}(x^2 - 25)^{3/2}\left[125 + 3(x^2 - 25)\right] + C$$
$$= \frac{1}{15}(x^2 - 25)^{3/2}(50 + 3x^2) + C$$

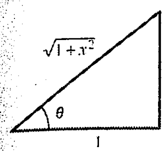
12. Same substitution as in Exercise 9

$$\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx = \int \frac{(5\sec\theta)^3}{5\tan\theta} 5\sec\theta\tan\theta\,d\theta$$
$$= 125\int \sec^4\theta\,d\theta$$
$$= 125\int (\tan^2\theta + 1)\sec^2\theta\,d\theta$$
$$= 125\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C$$
$$= \frac{125}{3}\frac{(x^2 - 25)^{3/2}}{125} + 125\frac{\sqrt{x^2 - 25}}{5} + C$$
$$= \frac{1}{3}(x^2 - 25)^{3/2} + 25(x^2 - 25)^{1/2} + C$$
$$= \frac{1}{3}\sqrt{x^2 - 25}(x^2 - 25 + 75) + C$$
$$= \frac{1}{3}\sqrt{x^2 - 25}(50 + x^2) + C$$

13. Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $\sqrt{1 + x^2} = \sec\theta$.

$$\int x\sqrt{1 + x^2}\, dx = \int \tan\theta(\sec\theta)\sec^2\theta\, d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}\left(1 + x^2\right)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.



14. Same substitution as in Exercise 13

$$\int \frac{9x^3}{\sqrt{1 + x^2}}\, dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\, d\theta = 9\int\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\, d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta\left(\sec^2\theta - 3\right) + C = 3\sqrt{1 + x^2}\left[\left(1 + x^2\right) - 3\right] + C = 3\sqrt{1 + x^2}\left(x^2 - 2\right) + C$$

15. Same substitution as in Exercise 13

$$\int \frac{1}{\left(1 + x^2\right)^2}\, dx = \int \frac{1}{\left(\sqrt{1 + x^2}\right)^4}\, dx = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\, d\theta = \frac{1}{2}\int(1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1 + x^2}}\right)\left(\frac{1}{\sqrt{1 + x^2}}\right)\right] + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{1 + x^2}\right) + C$$

16. Same substitution as in Exercise 13

$$\int \frac{x^2}{\left(1 + x^2\right)^2}\, dx = \int \frac{x^2}{\left(\sqrt{1 + x^2}\right)^4}\, dx = \int \frac{\tan^2\theta\sec^2\theta\, d\theta}{\sec^4\theta} = \int \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int(1 - \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1 + x^2}}\right)\left(\frac{1}{\sqrt{1 + x^2}}\right)\right] + C = \frac{1}{2}\left(\arctan x - \frac{x}{1 + x^2}\right) + C$$

17. Let $u = 4x$, $a = 3$, $du = 4\, dx$.

$$\int \sqrt{9 + 16x^2}\, dx = \frac{1}{4}\int \sqrt{(4x)^2 + 3^2}\,(4)\, dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\left[4x\sqrt{16x^2 + 9} + 9\ln\left|4x + \sqrt{16x^2 + 9}\right|\right] + C$$

$$= \frac{1}{2}x\sqrt{16x^2 + 9} + \frac{9}{8}\ln\left|4x + \sqrt{16x^2 + 9}\right| + C$$

18. Let $u = x, a = 2, du = dx$.

$$\int\sqrt{4 + x^2}\,dx = \int\sqrt{x^2 + 2^2}\,dx$$
$$= \frac{1}{2}\left[x\sqrt{x^2 + 4} + 4\ln\left|x + \sqrt{x^2 + 4}\right|\right] + C$$
$$= \frac{x}{2}\sqrt{x^2 + 4} + 2\ln\left|x + \sqrt{x^2 + 4}\right| + C$$

19. $$\int\sqrt{25 - 4x^2}\,dx = \int 2\sqrt{\frac{25}{4} - x^2}\,dx, \quad a = \frac{5}{2}$$
$$= 2\left(\frac{1}{2}\right)\left[\frac{25}{4}\arcsin\left(\frac{2x}{5}\right) + x\sqrt{\frac{25}{4} - x^2}\right] + C$$
$$= \frac{25}{4}\arcsin\left(\frac{2x}{5}\right) + \frac{x}{2}\sqrt{25 - 4x^2} + C$$

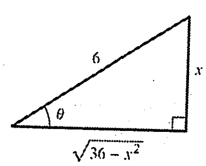
20. Let $u = \sqrt{5}x, a = 1, du = \sqrt{5}\,dx$.

$$\int\sqrt{5x^2 - 1}\,dx = \frac{1}{\sqrt{5}}\int\sqrt{\left(\sqrt{5}x\right)^2 - 1}\sqrt{5}\,dx$$
$$= \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)\left(\sqrt{5}x\sqrt{5x^2 - 1} - \ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right|\right) + C$$
$$= \frac{x}{2}\sqrt{5x^2 - 1} - \frac{\sqrt{5}}{10}\ln\left|\sqrt{5}x + \sqrt{5x^2 - 1}\right| + C$$

21. $$\int\frac{1}{\sqrt{16 - x^2}}\,dx = \arcsin\left(\frac{x}{4}\right) + C$$

22. Let $x = 6\sin\theta, dx = 6\cos\theta\,d\theta, \sqrt{36 - x^2} = 6\cos\theta$.

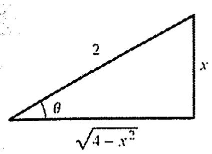
$$\int\frac{x^2}{\sqrt{36 - x^2}}\,dx = \int\frac{36\sin^2\theta}{6\cos\theta}(6\cos\theta\,d\theta)$$
$$= 36\int\sin^2\theta\,d\theta$$
$$= 18\int(1 - \cos 2\theta)\,d\theta$$
$$= 18\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$
$$= 18(\theta - \sin\theta\cos\theta) + C$$
$$= 18\left(\arcsin\frac{x}{6} - \frac{x}{6}\cdot\frac{\sqrt{36 - x^2}}{6}\right) + C$$
$$= 18\arcsin\frac{x}{6} - \frac{x\sqrt{36 - x^2}}{2} + C$$



23. Let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$,

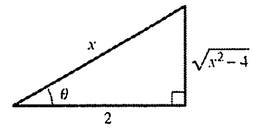
$\sqrt{4 - x^2} = 2\cos\theta$.

$$\begin{aligned}\int\sqrt{16 - 4x^2}\,dx &= 2\int\sqrt{4 - x^2}\,dx\\ &= 2\int 2\cos\theta(2\cos\theta\,d\theta)\\ &= 8\int\cos^2\theta\,d\theta\\ &= 4\int(1 + \cos 2\theta)\,d\theta\\ &= 4\left(\theta + \frac{1}{2}\sin 2\theta\right) + C\\ &= 4\theta + 4\sin\theta\cos\theta + C\\ &= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C\end{aligned}$$



24. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$,

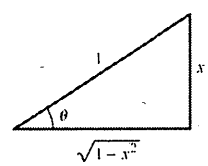
$\sqrt{x^2 - 4} = 2\tan\theta$.



$$\begin{aligned}\int\frac{1}{\sqrt{x^2 - 4}}\,dx &= \int\frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta\\ &= \int\sec\theta\,d\theta\\ &= \ln\left|\sec\theta + \tan\theta\right| + C\\ &= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C\\ &= \ln\left|x + \sqrt{x^2 - 4}\right| + C\end{aligned}$$

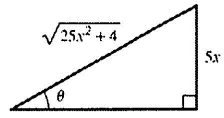
25. Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1 - x^2} = \cos\theta$.

$$\begin{aligned}\int\frac{\sqrt{1 - x^2}}{x^4}\,dx &= \int\frac{\cos\theta(\cos\theta\,d\theta)}{\sin^4\theta}\\ &= \int\cot^2\theta\csc^2\theta\,d\theta\\ &= -\frac{1}{3}\cot^3\theta + C\\ &= -\frac{\left(1 - x^2\right)^{3/2}}{3x^3} + C\end{aligned}$$



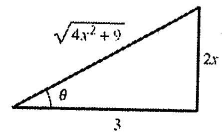
26. Let $5x = 2\tan\theta$, $5dx = 2\sec^2\theta\,d\theta$, $\sqrt{25x^2 + 4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta$.

$$\begin{aligned}\int\frac{\sqrt{25x^2 + 4}}{x^4}\,dx &= \int\frac{2\sec\theta}{\left(\frac{2}{5}\tan\theta\right)^4}\left(\frac{2}{5}\sec^2\theta\right)d\theta\\ &= \frac{125}{4}\int\frac{\cos\theta}{\sin^4\theta}\,d\theta\\ &= \frac{125}{4}\left(\frac{1}{(-3)\sin^3\theta}\right) + C\\ &= -\frac{125}{12}\csc^3\theta + C\\ &= -\frac{125}{12}\left(\frac{\sqrt{25x^2 + 4}}{5x}\right)^3\\ &= -\frac{\left(25x^2 + 4\right)^{3/2}}{12x^3} + C\end{aligned}$$



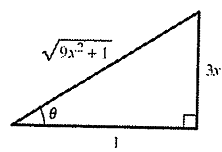
27. Let $2x = 3\tan\theta \Rightarrow x = \frac{3}{2}\tan\theta,\ dx = \frac{3}{2}\sec^2\theta\,d\theta,\ \sqrt{4x^2+9} = 3\sec\theta.$

$$\begin{aligned}\int\frac{1}{x\sqrt{4x^2+9}}\,dx &= \int\frac{(3/2)\sec^2\theta\,d\theta}{(3/2)\tan\theta\,3\sec\theta}\\ &= \frac{1}{3}\int\csc\theta\,d\theta\\ &= -\frac{1}{3}\ln\left|\csc\theta+\cot\theta\right|+C\\ &= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right|+C\end{aligned}$$



28. Let $3x = \tan\theta,\ 3dx = \sec^2\theta\,d\theta,\ \sqrt{9x^2+1} = \sec\theta.$

$$\begin{aligned}\int\frac{1}{x\sqrt{9x^2+1}}\,dx &= \int\frac{1}{\frac{1}{3}\tan\theta\sec\theta}\left(\frac{1}{3}\sec^2\theta\right)d\theta\\ &= \int\frac{\sec\theta}{\tan\theta}\,d\theta\\ &= \int\csc\theta\,d\theta\\ &= \ln\left|\csc\theta-\cot\theta\right|+C\\ &= \ln\left|\frac{\sqrt{9x^2+1}}{3x}-\frac{1}{3x}\right|+C\\ &= \ln\left|\frac{\sqrt{9x^2+1}-1}{3x}\right|+C\end{aligned}$$

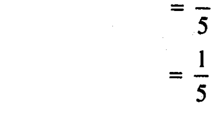


(**Note:** This equals $-\ln\left|\frac{\sqrt{9x^2+1}+1}{3x}\right|+C.$)

29. Let $u = x^2+3,\ du = 2x\,dx.$

$$\begin{aligned}\int\frac{-3x}{\left(x^2+3\right)^{3/2}}\,dx &= -\frac{3}{2}\int\left(x^2+3\right)^{-3/2}(2x)\,dx\\ &= -\frac{3}{2}\frac{\left(x^2+3\right)^{-1/2}}{(-1/2)}+C\\ &= \frac{3}{\sqrt{x^2+3}}+C\end{aligned}$$

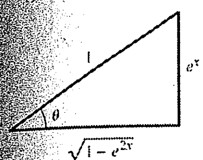
30. Let $x = \sqrt{5}\tan\theta,\ dx = \sqrt{5}\sec^2\theta,$
$x^2+5 = 5\sec^2\theta.$



$$\begin{aligned}\int\frac{1}{\left(x^2+5\right)^{3/2}}\,dx &= \int\frac{\sqrt{5}\sec^2\theta}{\left(\sqrt{5}\sec\theta\right)^3}\,d\theta\\ &= \frac{1}{5}\int\cos\theta\,d\theta\\ &= \frac{1}{5}\sin\theta+C = \frac{x}{5\sqrt{5+x^2}}+C\end{aligned}$$

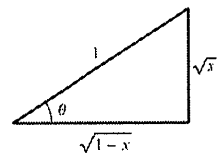
31. Let $e^x = \sin\theta$, $e^x\,dx = \cos\theta\,d\theta$, $\sqrt{1 - e^{2x}} = \cos\theta$.

$$\begin{aligned}\int e^x\sqrt{1 - e^{2x}}\,dx &= \int \cos^2\theta\,d\theta \\ &= \frac{1}{2}\int(1 + \cos 2\theta)\,d\theta \\ &= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) \\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1 - e^{2x}}\right) + C\end{aligned}$$



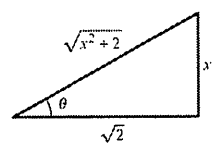
32. Let $\sqrt{x} = \sin\theta$, $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\sqrt{1 - x} = \cos\theta$.

$$\begin{aligned}\int\frac{\sqrt{1 - x}}{\sqrt{x}}\,dx &= \int\frac{\cos\theta(2\sin\theta\cos\theta\,d\theta)}{\sin\theta} \\ &= 2\int\cos^2\theta\,d\theta \\ &= \int(1 + \cos 2\theta)\,d\theta \\ &= (\theta + \sin\theta\cos\theta) + C \\ &= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1 - x} + C\end{aligned}$$



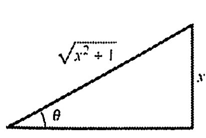
33. Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$, $x^2 + 2 = 2\sec^2\theta$.

$$\begin{aligned}\int\frac{1}{4 + 4x^2 + x^4}\,dx = \int\frac{1}{\left(x^2 + 2\right)^2}\,dx &= \int\frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta} \\ &= \frac{\sqrt{2}}{4}\int\cos^2\theta\,d\theta \\ &= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int(1 + \cos 2\theta)\,d\theta \\ &= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\ &= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{x^2 + 2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2 + 2}}\right) = \frac{1}{4}\left(\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$



34. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \sec^2\theta$.

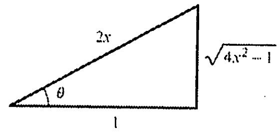
$$\begin{aligned}\int\frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx &= \frac{1}{4}\int\frac{4x^3 + 4x}{x^4 + 2x^2 + 1}\,dx + \int\frac{1}{\left(x^2 + 1\right)^2}\,dx \\ &= \frac{1}{4}\ln\left(x^4 + 2x^2 + 1\right) + \int\frac{\sec^2\theta\,d\theta}{\sec^4\theta} \\ &= \frac{1}{2}\ln\left(x^2 + 1\right) + \frac{1}{2}\int(1 + \cos 2\theta)\,d\theta \\ &= \frac{1}{2}\ln\left(x^2 + 1\right) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{2}\left(\ln\left(x^2 + 1\right) + \arctan x + \frac{x}{x^2 + 1}\right) + C\end{aligned}$$



35. Use integration by parts. Because $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x \Rightarrow du = \frac{1}{x\sqrt{4x^2 - 1}}\,dx,\ dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}}\,dx$$



$$2x = \sec\theta,\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2 - 1} = \tan\theta$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{\tan\theta} = x \operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\,d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C.$$

36. $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1 - x^2}}\,dx,\ dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1 - x^2}}\,dx$$

$$x = \sin\theta,\ dx = \cos\theta\,d\theta,\ \sqrt{1 - x^2} = \cos\theta$$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int (1 - \cos 2\theta)\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1 - x^2}\right) + C = \frac{1}{4}\left[\left(2x^2 - 1\right)\arcsin x + x\sqrt{1 - x^2}\right] + C$$

37. $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\,dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

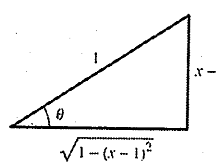
38. Let $x - 1 = \sin\theta,\ dx = \cos\theta\,d\theta,\ \sqrt{1 - (x - 1)^2} = \sqrt{2x - x^2} = \cos\theta.$

$$\int \frac{x^2}{\sqrt{2x - x^2}}\,dx = \int \frac{x^2}{\sqrt{1 - (x - 1)^2}}\,dx$$

$$= \int \frac{(1 + \sin\theta)^2(\cos\theta\,d\theta)}{\cos\theta}$$

$$= \int \left(1 + 2\sin\theta + \sin^2\theta\right)d\theta$$

$$= \int \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$



$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

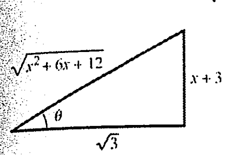
$$= \frac{3}{2}\arcsin(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + C$$

$$= \frac{3}{2}\arcsin(x - 1) - \frac{1}{2}\sqrt{2x - x^2}(x + 3) + C$$

39. $x^2 + 6x + 12 = x^2 + 6x + 9 + 3 = (x + 3)^2 + (\sqrt{3})^2$

Let $x + 3 = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$.

$$\sqrt{x^2 + 6x + 12} = \sqrt{(x + 3)^2 + (\sqrt{3})^2} = \sqrt{3}\sec\theta$$



$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 + 6x + 12}}\, dx &= \int \frac{\sqrt{3}\tan\theta - 3}{\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\, d\theta \\
&= \int \sqrt{3}\sec\theta\tan\theta\, d\theta - 3\int \sec\theta\, d\theta \\
&= \sqrt{3}\sec\theta - 3\ln\left|\sec\theta + \tan\theta\right| + C \\
&= \sqrt{3}\left(\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}}\right) - 3\ln\left|\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}} + \frac{x + 3}{\sqrt{3}}\right| + C \\
&= \sqrt{x^2 + 6x + 12} - 3\ln\left|\sqrt{x^2 + 6x + 12} + (x + 3)\right| + C
\end{aligned}$$

40. Let $x - 3 = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $\sqrt{(x - 3)^2 - 4} = 2\tan\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 - 6x + 5}}\, dx &= \int \frac{x}{\sqrt{(x - 3)^2 - 4}}\, dx \\
&= \int \frac{(2\sec\theta + 3)}{2\tan\theta}(2\sec\theta\tan\theta)\, d\theta \\
&= \int \left(2\sec^2\theta + 3\sec\theta\right) d\theta \\
&= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C_1 \\
&= 2\left[\frac{\sqrt{(x - 3)^3 - 4}}{2}\right] + 3\ln\left|\frac{x - 3}{2} + \frac{\sqrt{(x - 3)^2 - 4}}{2}\right| + C_1 \\
&= \sqrt{x^2 - 6x + 5} + 3\ln\left|(x - 3) + \sqrt{x^2 - 6x + 5}\right| + C
\end{aligned}$$

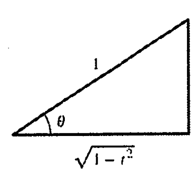
41. Let $t = \sin\theta$, $dt = \cos\theta\, d\theta$, $1 - t^2 = \cos^2\theta$.

(a) $\displaystyle\int \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = \int \tan^2\theta\, d\theta = \int\left(\sec^2\theta - 1\right) d\theta = \tan\theta - \theta + C = \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$

So, $\displaystyle\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$

(b) When $t = 0, \theta = 0$. When $t = \sqrt{3}/2, \theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{\left(1 - t^2\right)^{3/2}}\, dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$



42. Same substitution as in Exercise 41

(a) $$\int \frac{1}{(1-t^2)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int(\tan^2\theta + 1)\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + \frac{t}{\sqrt{1-t^2}} + C$$

So, $$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}}\right]_0^{\sqrt{3}/2} = \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When $t = 0, \theta = 0$. When $t = \sqrt{3}/2, \theta = \pi/3$. So,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}(\sqrt{3})^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

43. (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(27\tan^3\theta)(3\sec^2\theta\,d\theta)}{3\sec\theta}$$

$$= 27\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9\left[\sec^3\theta - 3\sec\theta\right] + C$$

$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

So, $$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$$

$$= \left(\frac{1}{3}(54\sqrt{2}) - 27\sqrt{2}\right) - (9 - 27) = 18 - 9\sqrt{2} = 9(2-\sqrt{2}) \approx 5.272.$$

(b) When $x = 0, \theta = 0$. When $x = 3, \theta = \pi/4$. So,

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1-3) = 9(2-\sqrt{2}) \approx 5.272.$$

44. (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\int\sqrt{9-25x^2}\,dx = \int(3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$

$$= \frac{9}{5}\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{10}(\theta + \sin\theta\cos\theta) + C$$

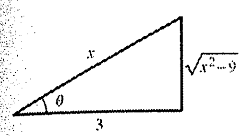
$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right) + C$$

So, $$\int_0^{3/5}\sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}.$$

(b) When $x = 0, \theta = 0$. When $x = \frac{3}{5}, \theta = \frac{\pi}{2}$.

So, $\int_0^{3/5} \sqrt{9 - 25x^2}\, dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}.$

45. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2 - 9} = 3\tan\theta$.



$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \int \frac{9\sec^2\theta}{3\tan\theta} 3\sec\theta\tan\theta\, d\theta \\
&= 9\int \sec^3\theta\, d\theta \\
&= 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right) \quad \text{(8.3 Exercise 102 or Example 5, Section 8.2)} \\
&= \frac{9}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) \\
&= \frac{9}{2}\left(\frac{x}{3} \cdot \frac{\sqrt{x^2 - 9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right|\right)
\end{aligned}$$

So,

$$\begin{aligned}
\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \frac{9}{2}\left[\frac{x\sqrt{x^2 - 9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right|\right]_4^6 \\
&= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right] \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left[\ln\left(\frac{6 + \sqrt{27}}{3}\right) - \ln\left(\frac{4 + \sqrt{7}}{3}\right)\right] \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right) \approx 12.644.
\end{aligned}$$

(b) When $x = 4, \theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$. When $x = 6, \theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

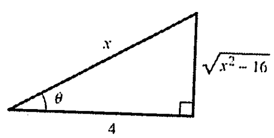
$$\begin{aligned}
\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\, dx &= \frac{9}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\
&= \frac{9}{2}\left(2 \cdot \sqrt{3} + \ln\left|2 + \sqrt{3}\right|\right) - \frac{9}{2}\left(\frac{4}{3}\left(\frac{\sqrt{7}}{3}\right) + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right) \\
&= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right) \approx 12.644
\end{aligned}$$

46. (a) Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-16} = 4\tan\theta$.

$$\begin{aligned}\int\frac{\sqrt{x^2-16}}{x^2}\,dx &= \int\frac{4\tan\theta}{16\sec^2\theta}(4\sec\theta\tan\theta)\,d\theta\\ &= \int\frac{\tan^2\theta}{\sec\theta}\,d\theta\\ &= \int\frac{\sin^2\theta}{\cos\theta}\,d\theta\\ &= \int\frac{1-\cos^2\theta}{\cos\theta}\,d\theta\\ &= \int\sec\theta\,d\theta - \int\cos\theta\,d\theta\\ &= \ln\left|\sec\theta+\tan\theta\right| - \sin\theta + C\\ &= \ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x} + C\end{aligned}$$

So,

$$\begin{aligned}\int_4^8\frac{\sqrt{x^2-16}}{x^2}\,dx &= \left[\ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x}\right]_4^8\\ &= \left[\ln\left(2+\frac{\sqrt{48}}{4}\right) - \frac{\sqrt{48}}{8}\right] - \left[\ln(1)\right]\\ &= \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}.\end{aligned}$$



(b) When $x = 4$, $\theta = 0$, and when $x = 8$, $\theta = \frac{\pi}{3}$. So,

$$\begin{aligned}\int_4^8\frac{\sqrt{x^2-16}}{x^2}\,dx &= \left[\ln\left|\sec\theta+\tan\theta\right| - \sin\theta\right]_0^{\pi/3}\\ &= \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.\end{aligned}$$

47. (a) Let $u = a\sin\theta$, $\sqrt{a^2-u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.

(b) Let $u = a\tan\theta$, $\sqrt{a^2+u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.

(c) Let $u = a\sec\theta$, $\sqrt{u^2-a^2} = \tan\theta$ if $u > a$ and $\sqrt{u^2-a^2} = -\tan\theta$ if $u < -a$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$.

48. (a) Substitution: $u = x^2 + 1$, $du = 2x\,dx$

(b) Trigonometric substitution: $x = \sec\theta$

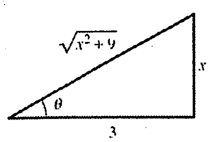
49. (a) $u = x^2 + 9, du = 2x\,dx$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+9) + C$$

Let $x = 3\tan\theta, x^2 + 9 = 9\sec^2\theta, dx = 3\sec^2\theta\,d\theta$.

$$\begin{aligned}\int \frac{x}{x^2+9}\,dx &= \int \frac{3\tan\theta}{9\sec^2\theta}3\sec^2\theta\,d\theta = \int \tan\theta\,d\theta \\ &= -\ln|\cos\theta| + C_1 \\ &= -\ln\left|\frac{3}{\sqrt{x^2+9}}\right| + C_1 \\ &= -\ln 3 + \ln\sqrt{x^2+9} + C_1 = \frac{1}{2}\ln(x^2+9) + C_2\end{aligned}$$

The answers are equivalent.



(b) $\int \frac{x^2}{x^2+9}\,dx = \int \frac{x^2+9-9}{x^2+9}\,dx = \int\left(1 - \frac{9}{x^2+9}\right)dx = x - 3\arctan\left(\frac{x}{3}\right) + C$

Let $x = 3\tan\theta, x^2 + 9 = 9\sec^2\theta, dx = 3\sec^2\theta\,d\theta$.

$$\begin{aligned}\int \frac{x^2}{x^2+9}\,dx &= \int \frac{9\tan^2\theta}{9\sec^2\theta}\,3\sec^2\theta\,d\theta \\ &= 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)\,d\theta \\ &= 3\tan\theta - 3\theta + C_1 \\ &= x - 3\arctan\left(\frac{x}{3}\right) + C_1\end{aligned}$$

The answers are equivalent.

50. (a) The graph of f is increasing when $f' > 0 : 0 < x < \infty$.

The graph of f is decreasing when $f' < 0 : -\infty < x < 0$.

(b) The graph of f is concave upward when the graph of f' is increasing. There are no such intervals.

The graph of f is concave downward when the graph of f' is decreasing:

$-\infty < x < 0$ and $0 < x < \infty$.

51. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \int d\theta$$

52. False

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}(\sec\theta\tan\theta\,d\theta) = \int \tan^2\theta\,d\theta$$

53. False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/3}\cos\theta\,d\theta$$

54. True

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx &= 2\int_0^1 x^2\sqrt{1-x^2}\,dx \\ &= 2\int_0^{\pi/2}(\sin^2\theta)(\cos\theta)(\cos\theta\,d\theta) \\ &= 2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\end{aligned}$$