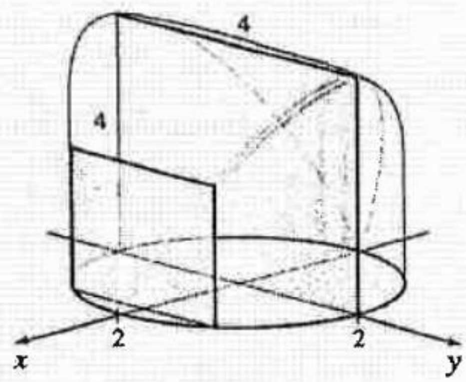
<u>AP Calculus – 8.4 – Volume by Cross Sections HW Solutions   pg. 610-612 #3, 5, 7, 14</u>

3. Find the volume of the solid whose base is a circle of radius of 2, if slices made perpendicular to the base are squares. See the figure.

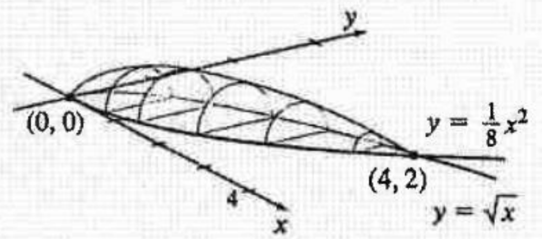


3. The equation of the circle is $x^2 + y^2 = 4$, so that for a given value of x, y can range from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$. Therefore, for each value of x from -2 to 2, the cross section is a square of side $2\sqrt{4-x^2}$, so that its area is $4(4-x^2)$. Therefore the volume of the solid is

$$V = \int_{-2}^{2} 4\left(4-x^2\right)\, dx = 4\int_{-2}^{2}\left(4-x^2\right)\, dx = 4\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \boxed{\frac{128}{3}}.$$

5. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are semicircles. See the figure.



(b) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are equilateral triangles.

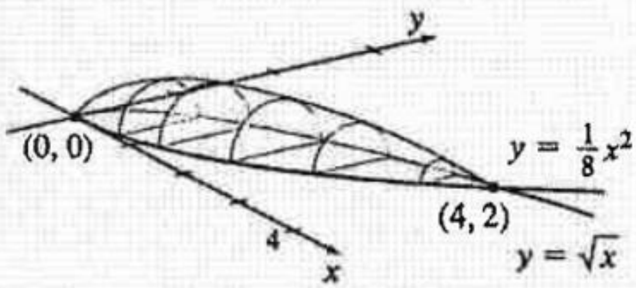
5. (a) From the figure, these two curves intersect at $(0,\ 0)$ and at $(4,\ 2)$, so we will integrate from 0 to 4. For each value of x, the cross section is a semicircle of diameter $\sqrt{x} - \frac{1}{8}x^2$, so its area is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{\sqrt{x}-\frac{1}{8}x^2}{2}\right)^2 = \frac{\pi}{512}\left(8\sqrt{x}-x^2\right)^2$$
$$= \frac{\pi}{512}\left(64x - 16x^{5/2} + x^4\right)$$

Therefore the volume is

$$V = \int_0^4 \frac{\pi}{512}\left(64x - 16x^{5/2} + x^4\right)dx = \frac{\pi}{512}\left[32x^2 - \frac{32}{7}x^{7/2} + \frac{1}{5}x^5\right]_0^4 = \boxed{\frac{9}{35}\pi}$$

5. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$.



(b) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are equilateral triangles.

(b) For each value of x, the cross section is an equilateral triangle with base $B = \sqrt{x} - \frac{1}{8}x^2$ and height $H = \frac{\sqrt{3}}{2}B = \frac{\sqrt{3}}{2}\left(\sqrt{x} - \frac{1}{8}x^2\right)$, so its area is

$$\begin{aligned} A &= \frac{1}{2}BH = \frac{1}{2}\left(\sqrt{x} - \frac{1}{8}x^2\right) \cdot \frac{\sqrt{3}}{2}\left(\sqrt{x} - \frac{1}{8}x^2\right) = \frac{\sqrt{3}}{4}\left(\sqrt{x} - \frac{1}{8}x^2\right)^2 \\ &= \frac{\sqrt{3}}{4}\left(x - \frac{1}{4}x^{5/2} + \frac{1}{64}x^4\right). \end{aligned}$$

Therefore the volume is

$$\begin{aligned} V &= \int_0^4 \frac{\sqrt{3}}{4}\left(x - \frac{1}{4}x^{5/2} + \frac{1}{64}x^4\right) dx = \frac{\sqrt{3}}{4}\left[\frac{1}{2}x^2 - \frac{1}{4}\cdot\frac{2}{7}x^{7/2} + \frac{1}{64}\cdot\frac{1}{5}x^5\right]_0^4 \\ &= \frac{\sqrt{3}}{4}\left[\left(\frac{1}{2}\cdot 4^2 - \frac{1}{14}\cdot 4^{7/2} + \frac{1}{320}\cdot 4^5\right) - 0\right] = \frac{\sqrt{3}}{4}\cdot\frac{72}{35} = \boxed{\frac{18\sqrt{3}}{35}} \end{aligned}$$

7. The base of a solid is the region bounded by the graphs of $y = x^2$, $x = 2$, and $y = 0$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis are squares.

(b) Find the volume V of the solid if slices made perpendicular to the x-axis are semicircles.

(c) Find the volume V of the solid if slices made perpendicular to the x-axis are equilateral triangles.

7. (a) For each value of x, the cross section is a square with side $s = y - 0 = x^2$, so its area is

$$A = s^2 = \left(x^2\right)^2 = x^4.$$

Therefore the volume is

$$V = \int_0^4 x^4\, dx = \left.\frac{x^5}{5}\right]_0^2 = \frac{2^5}{5} - 0 = \boxed{\frac{32}{5}}$$

(b) For each value of x, the cross section is a semicircle with diameter $D = x^2$ and therefore radius $r = \frac{D}{2} = \frac{x^2}{2}$, so its area is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot \left(\frac{x^2}{2}\right)^2 = \frac{\pi}{8}x^4.$$

Therefore the volume is

$$V = \frac{\pi}{8}\int_0^2 x^4\, dx = \frac{\pi}{8} \cdot \frac{32}{5} = \boxed{\frac{4}{5}\pi}$$

using the integral from Part (a).

(c) For each value of x, the cross section is a triangle of base $B = x^2$ and height $H = \frac{\sqrt{3}}{2}x^2$, so its area is

$$A = \frac{1}{2}BH = \frac{1}{2}x^2 \cdot \frac{\sqrt{3}}{2}x^2 = \frac{\sqrt{3}}{4}x^4.$$

Therefore the volume is

$$V = \frac{\sqrt{3}}{4}\int_0^2 x^4\, dx = \frac{\sqrt{3}}{4} \cdot \frac{32}{5} = \boxed{\frac{8\sqrt{3}}{5}}$$

using the integral from Part (a).

14. The base of a solid is the region enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$.

(a) Find the volume V of the solid if slices made perpendicular to the x-axis have cross sections that are triangles whose base is the distance between the graphs and whose height is three times the base.

(b) Find the volume V of the solid if slices made perpendicular to the y-axis have cross sections that are triangles whose base is the distance between the graphs and whose height is three times the base.

14. (a) Determine the point(s) of intersection of $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$:

$$\begin{aligned}
\sqrt{x} &= \frac{x^2}{8} \\
x &= \frac{x^4}{64} \\
64x &= x^4 \\
x^4 - 64x &= 0 \\
x\left(x^3 - 64\right) &= 0 \\
x\left(x-4\right)\left(x^2 + x + 16\right) &= 0 \\
x &= 0 \text{ or } x = 4
\end{aligned}$$

Then

$$\begin{aligned}
V &= \int_0^4 \frac{1}{2} \cdot 3 \left(\sqrt{x} - \frac{x^2}{8}\right)^2 dx \\
&= \frac{3}{2} \int_0^4 \left(x - \frac{x^2\sqrt{x}}{4} + \frac{x^4}{64}\right) dx \\
&= \frac{3}{2} \int_0^4 \left(x - \frac{x^{5/2}}{4} + \frac{x^4}{64}\right) dx \\
&= \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^{7/2}}{14} + \frac{x^5}{320}\right]_0^4 \\
&= \frac{3}{2} \left[8 - \frac{64}{7} + \frac{16}{5} - 0\right] \\
&= \boxed{\frac{108}{35}}
\end{aligned}$$

(b) Solve each equation for x as a function of y:

$$y = \sqrt{x}$$
$$x = y^2$$

$$y = \frac{x^2}{8}$$
$$x^2 = 8y$$
$$x = \sqrt{8y}$$

As determined above in 14(a), $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$ intersect at $x = 0$ and $x = 4$. The points of intersection are (0, 0) and (4, 2).

Then

$$\begin{aligned}
V &= \int_0^2 \frac{1}{2} \cdot 3 \left(\sqrt{8y} - y^2 \right)^2 dy \\
&= \frac{3}{2} \int_0^2 \left(8y - 4y^2\sqrt{2y} + y^4 \right) dy \\
&= \frac{3}{2} \int_0^2 \left(8y - 2^{5/2}y^{5/2} + y^4 \right) dy \\
&= \frac{3}{2} \left[4y^2 - \frac{2^{7/2}}{7} y^{7/2} + \frac{y^5}{5} \right]_0^2 \\
&= \frac{3}{2} \left[16 - \frac{128}{7} + \frac{32}{5} \right] \\
&= \boxed{\frac{216}{35}}
\end{aligned}$$