8.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Trigonometric Substitution In Exercises 1–4, state the trigonometric substitution you would use to find the indefinite integral. Do not integrate.

1. $\int (9 + x^2)^{-2}\, dx$
2. $\int \sqrt{4 - x^2}\, dx$
3. $\int \frac{x^2}{\sqrt{25 - x^2}}\, dx$
4. $\int x^2(x^2 - 25)^{3/2}\, dx$

Using Trigonometric Substitution In Exercises 5–8, find the indefinite integral using the substitution $x = 4 \sin \theta$.

5. $\int \frac{1}{(16 - x^2)^{3/2}}\, dx$
6. $\int \frac{4}{x^2\sqrt{16 - x^2}}\, dx$
7. $\int \frac{\sqrt{16 - x^2}}{x}\, dx$
8. $\int \frac{x^3}{\sqrt{16 - x^2}}\, dx$

Using Trigonometric Substitution In Exercises 9–12, find the indefinite integral using the substitution $x = 5 \sec \theta$.

9. $\int \frac{1}{\sqrt{x^2 - 25}}\, dx$
10. $\int \frac{\sqrt{x^2 - 25}}{x}\, dx$
11. $\int x^3\sqrt{x^2 - 25}\, dx$
12. $\int \frac{x^3}{\sqrt{x^2 - 25}}\, dx$

Using Trigonometric Substitution In Exercises 13–16, find the indefinite integral using the substitution $x = \tan \theta$.

13. $\int x\sqrt{1 + x^2}\, dx$
14. $\int \frac{9x^3}{\sqrt{1 + x^2}}\, dx$
15. $\int \frac{1}{(1 + x^2)^2}\, dx$
16. $\int \frac{x^2}{(1 + x^2)^2}\, dx$

Using Formulas In Exercises 17–20, use the Special Integration Formulas (Theorem 8.2) to find the indefinite integral.

17. $\int \sqrt{9 + 16x^2}\, dx$
18. $\int \sqrt{4 + x^2}\, dx$
19. $\int \sqrt{25 - 4x^2}\, dx$
20. $\int \sqrt{5x^2 - 1}\, dx$

Finding an Indefinite Integral In Exercises 21–36, find the indefinite integral.

21. $\int \frac{1}{\sqrt{16 - x^2}}\, dx$
22. $\int \frac{x^2}{\sqrt{36 - x^2}}\, dx$
23. $\int \sqrt{16 - 4x^2}\, dx$
24. $\int \frac{1}{\sqrt{x^2 - 4}}\, dx$
25. $\int \frac{\sqrt{1 - x^2}}{x^4}\, dx$
26. $\int \frac{\sqrt{25x^2 + 4}}{x^4}\, dx$
27. $\int \frac{1}{x\sqrt{4x^2 + 9}}\, dx$
28. $\int \frac{1}{x\sqrt{9x^2 + 1}}\, dx$
29. $\int \frac{-3x}{(x^2 + 3)^{3/2}}\, dx$
30. $\int \frac{1}{(x^2 + 5)^{3/2}}\, dx$
31. $\int e^x\sqrt{1 - e^{2x}}\, dx$
32. $\int \frac{\sqrt{1 - x}}{\sqrt{x}}\, dx$
33. $\int \frac{1}{4 + 4x^2 + x^4}\, dx$
34. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\, dx$
35. $\int \operatorname{arcsec} 2x\, dx, \quad x > \frac{1}{2}$
36. $\int x \arcsin x\, dx$

Completing the Square In Exercises 37–40, complete the square and find the indefinite integral.

37. $\int \frac{1}{\sqrt{4x - x^2}}\, dx$
38. $\int \frac{x^2}{\sqrt{2x - x^2}}\, dx$
39. $\int \frac{x}{\sqrt{x^2 + 6x + 12}}\, dx$
40. $\int \frac{x}{\sqrt{x^2 - 6x + 5}}\, dx$

Converting Limits of Integration In Exercises 41–46, evaluate the definite integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.

41. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\, dt$
42. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}}\, dt$
43. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}}\, dx$
44. $\int_0^{3/5} \sqrt{9 - 25x^2}\, dx$
45. $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\, dx$
46. $\int_4^8 \frac{\sqrt{x^2 - 16}}{x^2}\, dx$

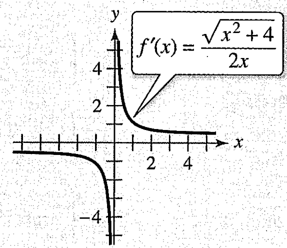
WRITING ABOUT CONCEPTS

47. **Trigonometric Substitution** State the substitution you would make if you used trigonometric substitution for an integral involving the given radical, where $a > 0$. Explain your reasoning.
   (a) $\sqrt{a^2 - u^2}$
   (b) $\sqrt{a^2 + u^2}$
   (c) $\sqrt{u^2 - a^2}$

48. **Choosing a Method** State the method of integration you would use to perform each integration. Explain why you chose that method. Do not integrate.
   (a) $\int x\sqrt{x^2 + 1}\, dx$ (b) $\int x^2\sqrt{x^2 - 1}\, dx$

49. **Comparing Methods**
   (a) Find the integral $\int \frac{x}{x^2 + 9}\, dx$ using u-substitution. Then find the integral using trigonometric substitution. Discuss the results.
   (b) Find the integral $\int \frac{x^2}{x^2 + 9}\, dx$ algebraically using $x^2 = (x^2 + 9) - 9$. Then find the integral using trigonometric substitution. Discuss the results.

50. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Identify the open interval(s) on which the graph of f is increasing or decreasing. Explain.

(b) Identify the open interval(s) on which the graph of f is concave upward or concave downward. Explain.

True or False? **In Exercises 51–54, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

51. If $x = \sin\theta$, then

$$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta.$$

52. If $x = \sec\theta$, then

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \sec\theta\tan\theta\,d\theta.$$

53. If $x = \tan\theta$, then

$$\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}} = \int_0^{4\pi/3} \cos\theta\,d\theta.$$

54. If $x = \sin\theta$, then

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta.$$

55. Area Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ shown in the figure.

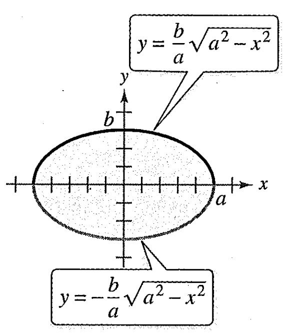


Figure for 55

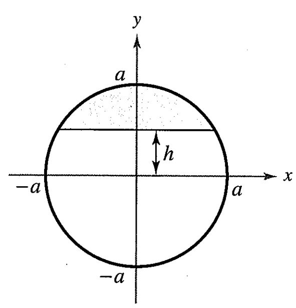
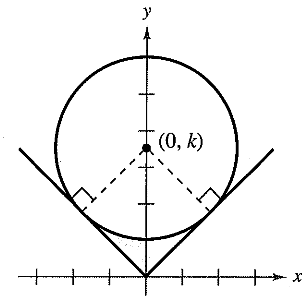


Figure for 56

56. Area Find the area of the shaded region of the circle of radius a when the chord is h units $(0 < h < a)$ from the center of the circle (see figure).

57. Mechanical Design The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y-k)^2 = 25$ (see figure).

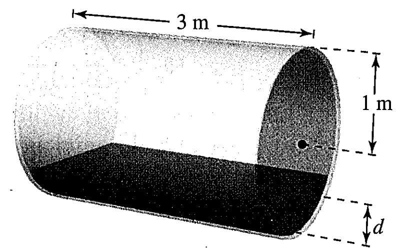


(a) Find k when the circle is tangent to the graph of $y = |x|$.

(b) Find the area of the surface of the machine part.

(c) Find the area of the surface of the machine part as a function of the radius r of the circle.

58. Volume The axis of a storage tank in the form of a right circular cylinder is horizontal (see figure). The radius and length of the tank are 1 meter and 3 meters, respectively.



(a) Determine the volume of fluid in the tank as a function of its depth d.

(b) Use a graphing utility to graph the function in part (a).

(c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

(d) Fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per minute. Determine the rate of change of the depth of the fluid as a function of its depth d.

(e) Use a graphing utility to graph the function in part (d). When will the rate of change of the depth be minimum? Does this agree with your intuition? Explain.

Volume of a Torus **In Exercises 59 and 60, find the volume of the torus generated by revolving the region bounded by the graph of the circle about the y-axis.**

59. $(x-3)^2 + y^2 = 1$

60. $(x-h)^2 + y^2 = r^2, \quad h > r$

Arc Length **In Exercises 61 and 62, find the arc length of the curve over the given interval.**

61. $y = \ln x, \quad [1, 5]$

62. $y = \frac{1}{2}x^2, \quad [0, 4]$

63. Arc Length Show that the length of one arch of the sine curve is equal to the length of one arch of the cosine curve.

64. Conjecture

(a) Find formulas for the distances between $(0, 0)$ and (a, a^2) along the line between these points and along the parabola $y = x^2$.

(b) Use the formulas from part (a) to find the distances for $a = 1$ and $a = 10$.

(c) Make a conjecture about the difference between the two distances as a increases.

Centroid In Exercises 65 and 66, find the centroid of the region determined by the graphs of the inequalities.

65. $y \le 3/\sqrt{x^2 + 9},\ y \ge 0,\ x \ge -4,\ x \le 4$

66. $y \le \frac{1}{4}x^2,\ (x - 4)^2 + y^2 \le 16,\ y \ge 0$

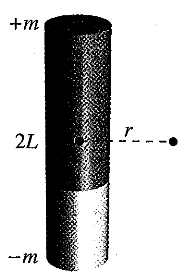
67. Surface Area Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis.

68. Field Strength The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

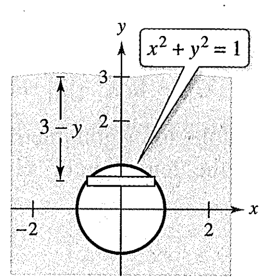
where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\, dr.$$



69. Fluid Force

Find the fluid force on a circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery when the center of the window is (a) 3 feet and (b) d feet $(d > 1)$ below the water's surface (see figure). Use trigonometric substitution to evaluate the one integral. Water weighs 62.4 pounds per cubic foot. (Recall that in Section 7.7 in a similar problem, you evaluated one integral by a geometric formula and the other by observing that the integrand was odd.)



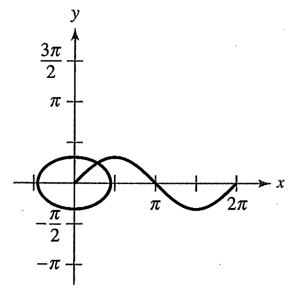
70. Fluid Force Evaluate the following two integrals, which yield the fluid forces given in Example 6.

(a) $F_{\text{inside}} = 48\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\, dy$

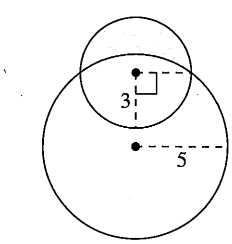
(b) $F_{\text{outside}} = 64\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\, dy$

71. Verifying Formulas Use trigonometric substitution to verify the integration formulas given in Theorem 8.2.

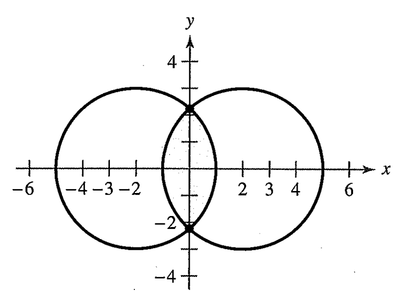
72. Arc Length Show that the arc length of the graph of $y = \sin x$ on the interval $[0, 2\pi]$ is equal to the circumference of the ellipse $x^2 + 2y^2 = 2$ (see figure).



73. Area of a Lune The crescent-shaped region bounded by two circles forms a *lune* (see figure). Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.



74. Area Two circles of radius 3, with centers at $(-2, 0)$ and $(2, 0)$, intersect as shown in the figure. Find the area of the shaded region.



PUTNAM EXAM CHALLENGE

75. Evaluate

$$\int_0^1 \frac{\ln(x + 1)}{x^2 + 1}\, dx.$$

This problem was composed by the Committee on the Putnam Prize Competition.