75. Let $I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

Let $x = \frac{1-u}{1+u}, \quad dx = \frac{-2}{(1+u)^2}\,du$

$x + 1 = \frac{2}{1+u}, \quad x^2 + 1 = \frac{2+2u^2}{(1+u)^2}$

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\left(\frac{2+2u^2}{(1+u)^2}\right)}\left(\frac{-2}{(1+u)^2}\right)du$$

$$= \int_1^0 \frac{-\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1+u^2}\,du = \int_0^1 \frac{\ln 2}{1+u^2} - \int_0^1 \frac{\ln(1+u)}{1+u^2}\,du = (\ln 2)\left[\arctan u\right]_0^1 - I$$

$$\Rightarrow 2I = \ln 2\left(\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{8}\ln 2 \approx 0.272198$$

Section 8.5 Partial Fractions

1. $\frac{4}{x^2 - 8x} = \frac{4}{x(x-8)} = \frac{A}{x} + \frac{B}{x-8}$

2. $\frac{2x^2+1}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

3. $\frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

4. $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

5. $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x+3} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+3)$

When $x = 3$, $1 = 6B \Rightarrow B = \frac{1}{6}$.

When $x = -3$, $1 = -6A \Rightarrow A = -\frac{1}{6}$.

$$\int \frac{1}{x^2-9}\,dx = -\frac{1}{6}\int \frac{1}{x+3}\,dx + \frac{1}{6}\int \frac{1}{x-3}\,dx$$
$$= -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + C$$
$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

6. $\frac{2}{9x^2-1} = \frac{2}{(3x-1)(3x+1)} = \frac{A}{3x-1} + \frac{B}{3x+1}$

$2 = A(3x+1) + B(3x-1)$

When $x = \frac{1}{3}$, $2 = 2A \Rightarrow A = 1$.

When $x = -\frac{1}{3}$, $2 = -2B \Rightarrow B = -1$.

$$\int \frac{2}{9x^2-1}\,dx = \int \frac{1}{3x-1}\,dx + \int \frac{-1}{3x+1}\,dx$$
$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$
$$= \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

7. $\frac{5}{x^2+3x-4} = \frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$5 = A(x-1) + B(x+4)$

When $x = 1$, $5 = 5B \Rightarrow B = 1$.

When $x = -4$, $5 = -5A \Rightarrow A = -1$.

$$\int \frac{5}{x^2+3x-4}\,dx = \int \frac{-1}{x+4}\,dx + \int \frac{1}{x-1}\,dx$$
$$= -\ln|x+4| + \ln|x-1| + C$$
$$= \ln\left|\frac{x-1}{x+4}\right| + C$$

8. $\dfrac{3-x}{3x^2-2x-1} = \dfrac{3-x}{(3x+1)(x-1)} = \dfrac{A}{3x+1} + \dfrac{B}{x-1}$

$3 - x = A(x-1) + B(3x+1)$

When $x = 1$, $2 = 4B \Rightarrow B = \dfrac{1}{2}$.

When $x = -\dfrac{1}{3}$, $\dfrac{10}{3} = -\dfrac{4}{3}A \Rightarrow A = -\dfrac{5}{2}$.

$$\int \frac{3-x}{3x^2-2x-1}\,dx = -\frac{5}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$
$$= \frac{-5}{6}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C$$

9. $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

When $x = 0$, $12 = -4A \Rightarrow A = -3$.

When $x = -2$, $-8 = 8B \Rightarrow B = -1$.

When $x = 2$, $40 = 8C \Rightarrow C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx = 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

10. $\dfrac{x^3-x+3}{x^2+x-2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$2x + 1 = A(x-1) + B(x+2)$

When $x = -2$, $-3 = -3A \Rightarrow A = 1$.

When $x = 1$, $3 = 3B \Rightarrow B = 1$.

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int\left(x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right)dx = \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2+x-2| + C$$

11. $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$x + 5 = A(x+2) + B(x-4)$

When $x = 4$, $9 = 6A \Rightarrow A = \dfrac{3}{2}$.

When $x = -2$, $3 = -6B \Rightarrow B = -\dfrac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int\left(2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right)dx = x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

12. $\dfrac{x+2}{x^2+5x} = \dfrac{x+2}{x(x+5)} = \dfrac{A}{x} + \dfrac{B}{x+5}$

$x + 2 = A(x+5) + Bx$

When $x = -5$, $-3 = -5B \Rightarrow B = \dfrac{3}{5}$.

When $x = 0$, $2 = 5A \Rightarrow A = \dfrac{2}{5}$.

$$\int \frac{x+2}{x^2+5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$
$$= \frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C$$

13. $\dfrac{4x^2+2x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $B = -1$.

When $x = -1$, $C = 1$.

When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int\left(\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$
$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

14. $\dfrac{5x-2}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$

$5x - 2 = A(x-2) + B$

When $x = 2$, $8 = B$.

When $x = 0$, $-2 = -2A + B = -2A + 8 \Rightarrow A = 5$.

$$\int \frac{5x-2}{(x-2)^2}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{8}{(x-2)^2}\,dx$$
$$= 5\ln|x-2| - \frac{8}{x-2} + C$$

15. $\dfrac{x^2+3x-4}{x^3-4x^2+4x} = \dfrac{x^2+3x-4}{x(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$

When $x = 0$, $-4 = 4A \Rightarrow A = -1$.

When $x = 2$, $6 = 2C \Rightarrow C = 3$.

When $x = 1$, $0 = -1 - B + 3 \Rightarrow B = 2$.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{(x-2)}\,dx + \int \frac{3}{(x-2)^2}\,dx = -\ln|x| + 2\ln|x-2| - \frac{3}{(x-2)} + C$$

16. $\dfrac{8x}{x^3+x^2-x-1} = \dfrac{8x}{x^2(x+1)-(x+1)} = \dfrac{8x}{(x+1)(x-1)(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$8x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

When $x = 1$, $8 = 4A \Rightarrow A = 2$.

When $x = -1$, $-8 = -2C \Rightarrow C = 4$.

When $x = 0$, $0 = A - B - C = 2 - B - 4 \Rightarrow B = -2$.

$$\int \frac{8x}{x^3+x^2-x-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-2}{x+1}\,dx + \int \frac{4}{(x+1)^2}\,dx$$
$$= 2\ln|x-1| - 2\ln|x+1| - \frac{4}{x+1} + C$$

17. $\dfrac{x^2 - 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0, A = -1$.

When $x = 1, 0 = -2 + B + C$.

When $x = -1, 0 = -2 + B - C$.

Solving these equations you have $A = -1, B = 2, C = 0$.

$$\int \frac{x^2 - 1}{x^3 + x}\,dx = -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2 + 1}\,dx = -\ln|x| + \ln|x^2 + 1| + C = \ln\left|\frac{x^2 + 1}{x}\right| + C$$

18. $\dfrac{6x}{x^3 - 8} = \dfrac{6x}{(x - 2)(x^2 + 2x + 4)} = \dfrac{A}{x - 2} + \dfrac{Bx + C}{x^2 + 2x + 4}$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x - 2)$$

When $x = 2, 12 = 12A \Rightarrow A = 1$.

When $x = 0, 0 = 4 - 2C \Rightarrow C = 2$.

When $x = 1, 6 = 7 + (B + 2)(-1) \Rightarrow B = -1$.

$$\begin{aligned}\int \frac{6x}{x^3 - 8}\,dx &= \int \frac{1}{x - 2}\,dx + \int \frac{-x + 2}{x^2 + 2x + 4}\,dx \\ &= \int \frac{1}{x - 2}\,dx + \int \frac{-x - 1}{x^2 + 2x + 4}\,dx + \int \frac{3}{(x^2 + 2x + 1) + 3}\,dx \\ &= \ln|x - 2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x + 1}{\sqrt{3}}\right) + C \\ &= \ln|x - 2| - \frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x + 1)}{3}\right) + C\end{aligned}$$

19. $\dfrac{x^2}{x^4 - 2x^2 - 8} = \dfrac{A}{x - 2} + \dfrac{B}{x + 2} + \dfrac{Cx + D}{x^2 + 2}$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2, 4 = 24A$.

When $x = -2, 4 = -24B$.

When $x = 0, 0 = 4A - 4B - 4D$.

When $x = 1, 1 = 9A - 3B - 3C - 3D$.

Solving these equations you have $A = \frac{1}{6}, B = -\frac{1}{6}, C = 0, D = \frac{1}{3}$.

$$\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx = \frac{1}{6}\left(\int \frac{1}{x - 2}\,dx - \int \frac{1}{x + 2}\,dx + 2\int \frac{1}{x^2 + 2}\,dx\right) = \frac{1}{6}\left(\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right) + C$$

20. $\dfrac{x}{(2x-1)(2x+1)(4x^2+1)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1} + \dfrac{Cx+D}{4x^2+1}$

$$x = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

When $x = \frac{1}{2}, \frac{1}{2} = 4A.$

When $x = -\frac{1}{2}, -\frac{1}{2} = -4B.$

When $x = 0, 0 = A - B - D.$

When $x = 1, 1 = 15A + 5B + 3C + 3D.$

Solving these equations you have $A = \frac{1}{8}, B = \frac{1}{8}, C = -\frac{1}{2}, D = 0.$

$$\int \frac{x}{16x^4-1}\,dx = \frac{1}{8}\left(\int \frac{1}{2x-1}\,dx + \int \frac{1}{2x+1}\,dx - 4\int \frac{x}{4x^2+1}\,dx\right) = \frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + C$$

21. $\dfrac{x^2+5}{(x+1)(x^2-2x+3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-2x+3}$

$$x^2 + 5 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$
$$= (A + B)x^2 + (-2A + B + C)x + (3A + C)$$

When $x = -1, A = 1.$

By equating coefficients of like terms, you have $A + B = 1, -2A + B + C = 0, 3A + C = 5.$

Solving these equations you have $A = 1, B = 0, C = 2.$

$$\int \frac{x^2+5}{x^3-x^2+x+3}\,dx = \int \frac{1}{x+1}\,dx + 2\int \frac{1}{(x-1)^2+2}\,dx = \ln|x+1| + \sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

22. $\dfrac{x^2+6x+4}{x^4+8x^2+16} = \dfrac{x^2+6x+4}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$

$$x^2 + 6x + 4 = (Ax + B)(x^2 + 4) + Cx + D$$
$$= Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

By equating coefficients of like terms, you have

$A = 0, \quad B = 1, \quad 4A + C = 6, \quad 4B + D = 4.$

Solving these equations you have $A = 0, B = 1, C = 6, D = 0.$

$$\int \frac{x^2+6x+4}{x^4+8x^2+16}\,dx = \int \frac{1}{x^2+4}\,dx + \int \frac{6x}{(x^2+4)^2}\,dx$$
$$= \frac{1}{2}\arctan\frac{x}{2} - \frac{3}{x^2+4} + C$$

23. $\dfrac{3}{4x^2 + 5x + 1} = \dfrac{3}{(4x + 1)(x + 1)} = \dfrac{A}{4x + 1} + \dfrac{B}{x + 1}$

$3 = A(x + 1) + B(4x + 1)$

When $x = -1, 3 = -3B \Rightarrow B = -1$.

When $-\frac{1}{4}, 3 = \frac{3}{4}A \Rightarrow A = 4$.

$$\int_0^2 \frac{3}{4x^2 + 5x + 1}\,dx = \int_0^2 \frac{4}{4x + 1}\,dx + \int_0^2 \frac{-1}{x + 1}\,dx$$
$$= \Big[\ln|4x + 1| - \ln|x + 1|\Big]_0^2$$
$$= \ln 9 - \ln 3$$
$$= 2\ln 3 - \ln 3 = \ln 3$$

24. $\dfrac{x - 1}{x^2(x + 1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x + 1}$

$x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$

When $x = 0, B = -1$.

When $x = -1, C = -2$.

When $x = 1, 0 = 2A + 2B + C$.

Solving these equations you have
$A = 2, B = -1, C = -2$.

$$\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx = 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x + 1}\,dx$$
$$= \left[2\ln|x| + \frac{1}{x} - 2\ln|x + 1|\right]_1^5$$
$$= \left[2\ln\left|\frac{x}{x + 1}\right| + \frac{1}{x}\right]_1^5$$
$$= 2\ln\frac{5}{3} - \frac{4}{5}$$

25. $\dfrac{x + 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

$x + 1 = A(x^2 + 1) + (Bx + C)x$

When $x = 0, A = 1$.

When $x = 1, 2 = 2A + B + C$.

When $x = -1, 0 = 2A + B - C$.

Solving these equations we have
$A = 1, B = -1, C = 1$.

$$\int_1^2 \frac{x + 1}{x(x^2 + 1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2 + 1}\,dx + \int_1^2 \frac{1}{x^2 + 1}\,dx$$
$$= \left[\ln|x| - \frac{1}{2}\ln(x^2 + 1) + \arctan x\right]_1^2$$
$$= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2$$
$$\approx 0.557$$

26. $$\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx = \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1}\,dx$$
$$= \Big[x - \ln|x^2 + x + 1|\Big]_0^1$$
$$= 1 - \ln 3$$

27. Let $u = \cos x$, $du = -\sin x\,dx$.

$\dfrac{1}{u(u + 1)} = \dfrac{A}{u} + \dfrac{B}{u + 1}$

$1 = A(u + 1) + Bu$

When $u = 0, A = 1$.

When $u = -1, B = -1$.

$$\int \frac{\sin x}{\cos x + \cos^2 x}\,dx = -\int \frac{1}{u(u + 1)}\,du$$
$$= \int \frac{1}{u + 1}\,du - \int \frac{1}{u}\,du$$
$$= \ln|u + 1| - \ln|u| + C$$
$$= \ln\left|\frac{u + 1}{u}\right| + C$$
$$= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C$$
$$= \ln|1 + \sec x| + C$$

28. $$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx = 5\int \frac{1}{u^2 + 3u - 4}\,du$$
$$= \ln\left|\frac{u - 1}{u + 4}\right| + C$$
$$= \ln\left|\frac{-1 + \sin x}{4 + \sin x}\right| + C$$

(From Exercise 7 with $u = \sin x$, $du = \cos x\,dx$)

29. Let $u = \tan x$, $du = \sec^2 x\,dx$.

$\dfrac{1}{u^2 + 5u + 6} = \dfrac{1}{(u + 3)(u + 2)} = \dfrac{A}{u + 3} + \dfrac{B}{u + 2}$

$1 = A(u + 2) + B(u + 3)$

When $u = -2, 1 = B$.

When $u = -3, 1 = -A \Rightarrow A = -1$.

$$\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx = \int \frac{1}{u^2 + 5u + 6}\,du$$
$$= \int \frac{-1}{u + 3}\,du + \int \frac{1}{u + 2}\,du$$
$$= -\ln|u + 3| + \ln|u + 2| + C$$
$$= \ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$$

30. $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1}$, $u = \tan x$, $du = \sec^2 x\,dx$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\int \frac{\sec^2 x\,dx}{\tan x(\tan x + 1)} = \int \frac{1}{u(u+1)}\,du$$
$$= \int\left(\frac{1}{u} - \frac{1}{u+1}\right)du$$
$$= \ln|u| - \ln|u+1| + C$$
$$= \ln\left|\frac{u}{u+1}\right| + C$$
$$= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C$$

31. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$
$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \dfrac{1}{5}$.

When $u = -4$, $B = -\dfrac{1}{5}$.

$$\int \frac{e^x}{(e^x-1)(e^x+4)}\,dx = \int \frac{1}{(u-1)(u+4)}\,du$$
$$= \frac{1}{5}\left(\int \frac{1}{u-1}\,du - \int \frac{1}{u+4}\,du\right)$$
$$= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C$$
$$= \frac{1}{5}\ln\left|\frac{e^x-1}{e^x+4}\right| + C$$

32. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+1}$$
$$1 = A(u^2+1) + (Bu+C)(u-1)$$

When $u = 1$, $A = \dfrac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$.

Solving these equations you have $A = \dfrac{1}{2}$, $B = -\dfrac{1}{2}$, and $C = -\dfrac{1}{2}$.

$$\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx = \int \frac{1}{(u^2+1)(u-1)}\,du$$
$$= \frac{1}{2}\left(\int \frac{1}{u-1}\,du - \int \frac{u+1}{u^2+1}\,du\right)$$
$$= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C$$
$$= \frac{1}{4}\left(2\ln|e^x-1| - \ln|e^{2x}+1| - 2\arctan e^x\right) + C$$

33. Let $u = \sqrt{x}, u^2 = x, 2u\,du = dx$.

$$\int \frac{\sqrt{x}}{x-4}\,dx = \int \frac{u(2u)du}{u^2-4} = \int\left(\frac{2u^2-8}{u^2-4} + \frac{8}{u^2-4}\right)du = \int\left(2 + \frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4} = \frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$8 = A(u+2) + B(u-2)$$

When $u = -2, 8 = -4B \Rightarrow B = -2$.

When $u = 2, 8 = 4A \Rightarrow A = 2$.

$$\int\left(2 + \frac{8}{u^2-4}\right)du = 2u + \int\left(\frac{2}{u-2} - \frac{2}{u+2}\right)du$$

$$= 2u + 2\ln|u-2| - 2\ln|u+2| + C$$

$$= 2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$$

34. Let $u = x^{1/6}, u^2 = x^{1/3}, u^3 = x^{1/2}, u^6 = x, 6u^5\,du = dx$.

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx = \int \frac{6u^5\,du}{u^3-u^2} = 6\int\frac{u^3\,du}{u-1}$$

$$= 6\int\left(u^2 + u + 1 + \frac{1}{u-1}\right)du \qquad \text{(long division)}$$

$$= 6\left(\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right) + C$$

$$= 2\sqrt{x} + 3x^{1/3} + 6x^{1/6} + 6\ln\left|x^{1/6} - 1\right| + C$$

35. $$\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When $x = 0, 1 = aA \Rightarrow A = 1/a$.

When $x = -a/b, 1 = -(a/b)B \Rightarrow B = -b/a$.

$$\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a+bx}\right)dx$$

$$= \frac{1}{a}\left(\ln|x| - \ln|a+bx|\right) + C$$

$$= \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$$

36. $$\frac{1}{a^2-x^2} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$1 = A(a+x) + B(a-x)$$

When $x = a, 1 = 2aA \Rightarrow A = 1/2a$.

When $x = -a, 1 = 2aB \Rightarrow B = 1/2a$.

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx$$

$$= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

37. $\dfrac{x}{(a+bx)^2} = \dfrac{A}{a+bx} + \dfrac{B}{(a+bx)^2}$

$x = A(a+bx) + B$

When $x = -a/b$, $B = -a/b$.

When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\begin{aligned}\int \frac{x}{(a+bx)^2}\,dx &= \int\left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2}\right)dx\\ &= \frac{1}{b}\int\frac{1}{a+bx}\,dx - \frac{a}{b}\int\frac{1}{(a+bx)^2}\,dx\\ &= \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\left(\frac{1}{a+bx}\right) + C\\ &= \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C\end{aligned}$$

38. $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$1 = Ax(a+bx) + B(a+bx) + Cx^2$

When $x = 0$, $1 = Ba \Rightarrow B = 1/a$. When $x = -a/b$,

$1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$. When $x = 1$,

$1 = (a+b)A + (a+b)B + C \Rightarrow A = -b/a^2$.

$$\begin{aligned}\int\frac{1}{x^2(a+bx)}\,dx &= \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx\\ &= -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C\\ &= -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C\\ &= -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C\end{aligned}$$

39. Dividing x^3 by $x - 5$

40. (a) $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$

(b) $\dfrac{N(x)}{D(x)} = \dfrac{A_1 + B_1x}{(ax^2+bx+c)} + \cdots + \dfrac{A_n + B_nx}{(ax^2+bx+c)^n}$

41. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

(c) Trigonometric substitution (tan) or inverse tangent rule

42. (a) Yes. Because $f' > 0$ on $(0, 5)$, f is increasing, and $f(3) > f(2)$. Therefore, $f(3) - f(2) > 0$.

(b) The area under the graph of f' is greater on the interval $[1, 2]$ because the graph is decreasing on $[1, 4]$.

43. $A = \displaystyle\int_0^1 \frac{12}{x^2+5x+6}\,dx$

$\dfrac{12}{x^2+5x+6} = \dfrac{12}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$12 = A(x+3) + B(x+2)$

Let $x = -3$: $12 = B(-1) \Rightarrow B = -12$

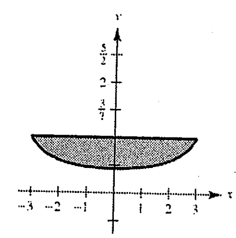
Let $x = -2$: $12 = A(1) \Rightarrow A = 12$

$$\begin{aligned}A &= \int_0^1\left(\frac{12}{x+2} - \frac{12}{x+3}\right)dx\\ &= \Big[12\ln|x+2| - 12\ln|x+3|\Big]_0^1\\ &= 12(\ln 3 - \ln 4 - \ln 2 + \ln 3)\\ &= 12\ln\left(\frac{9}{8}\right) \approx 1.4134\end{aligned}$$

44. $A = 2\displaystyle\int_0^3\left(1 - \frac{7}{16-x^2}\right)dx = 2\int_0^3 dx - 14\int_0^3\frac{1}{16-x^2}\,dx$

$= \left[2x - \dfrac{14}{8}\ln\left|\dfrac{4+x}{4-x}\right|\right]_0^3$ (From Exercise 36)

$= 6 - \dfrac{7}{4}\ln 7 \approx 2.595$

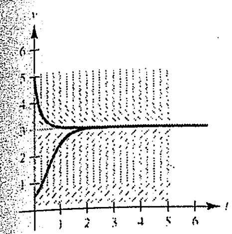


45. Average cost $= \dfrac{1}{80-75}\displaystyle\int_{75}^{80}\frac{124p}{(10+p)(100-p)}\,dp$

$$\begin{aligned}&= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10+p)11} + \frac{1240}{(100-p)11}\right)dp\\ &= \frac{1}{5}\left[\frac{-124}{11}\ln(10+p) - \frac{1240}{11}\ln(100-p)\right]_{75}^{80}\\ &\approx \frac{1}{5}(24.51) = 4.9\end{aligned}$$

Approximately \$490,000

46. (a)



(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t\to\infty} y(t) = 3$.

(d) $$\frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \Rightarrow A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{L}\left[\int \frac{1}{y}\,dy + \int \frac{1}{L-y}\,dy\right] = \int k\,dt$$

$$\frac{1}{L}\left[\ln|y| - \ln|L-y|\right] = kt + C_1$$

$$\ln\left|\frac{y}{L-y}\right| = kLt + LC_1$$

$$C_2e^{kLt} = \frac{y}{L-y}$$

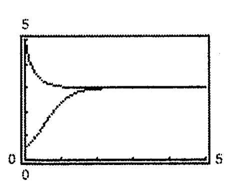
When $t = 0$, $\frac{y_0}{L-y_0} = C_2 \Rightarrow \frac{y}{L-y} = \frac{y_0}{L-y_0}e^{kLt}$.

Solving for y, you obtain $y = \frac{y_0L}{y_0 + (L-y_0)e^{-kLt}}$.

(e) $k = 1, L = 3$

(*i*) $y(0) = 5$: $y = \frac{15}{5 - 2e^{-3t}}$

(*ii*) $y(0) = \frac{1}{2}$: $y = \frac{3/2}{(1/2) + (5/2)e^{-3t}} = \frac{3}{1 + 5e^{-3t}}$



(f) $$\frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k\left[y\left(\frac{-dy}{dt}\right) + (L-y)\frac{dy}{dt}\right] = 0$$

$$\Rightarrow \quad y\frac{dy}{dt} = (L-y)\frac{dy}{dt}$$

$$\Rightarrow \quad y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

47. $V = \pi \int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi \int_0^3 \frac{x^2}{\left(x^2+1\right)^2} dx$

$= 4\pi \int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right) dx$ (partial fractions)

$= 4\pi \left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3$ (trigonometric substitution)

$= 2\pi \left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$

$A = \int_0^3 \frac{2x}{x^2+1} dx = \left[\ln\left(x^2+1\right)\right]_0^3 = \ln 10$

$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2+1} dx = \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2+1}\right) dx = \frac{1}{\ln 10}\left[2x - 2\arctan x\right]_0^3 = \frac{2}{\ln 10}(3 - \arctan 3) \approx 1.521$

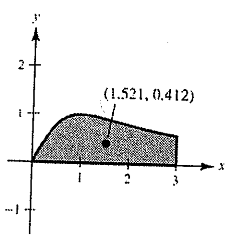
$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{\left(x^2+1\right)^2} dx$

$= \frac{2}{\ln 10}\int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right) dx$ (partial fractions)

$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3$ (trigonometric substitution)

$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2\left(x^2+1\right)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right) \approx 0.412$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$



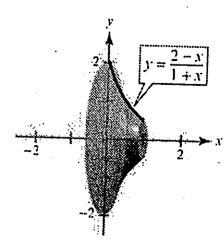
48. $y^2 = \frac{(2-x)^2}{(1+x)^2}, \quad [0, 1]$

$V = \int_0^1 \pi \frac{(2-x)^2}{(1+x)^2} dx$

$= \pi\left[\int_0^1 \frac{4}{(1+x)^2} dx - \int_0^1 \frac{4x}{(1+x)^2} dx + \int_0^1 \frac{x^2}{(1+x)^2} dx\right]$

$= \pi\left[2 - (4\ln 2 - 2) + \frac{3}{2} - 2\ln 2\right]$

$= \pi\left(\frac{11}{2} - 6\ln 2\right) = \frac{\pi}{2}(11 - 12\ln 2)$



49. $\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, A = B = \frac{1}{n+1}$

$$\frac{1}{n+1}\int\left(\frac{1}{x+1} + \frac{1}{n-x}\right)dx = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

When $t = 0, x = 0, C = \frac{1}{n+1}\ln\frac{1}{n}$.

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\frac{1}{n}$$

$$\frac{1}{n+1}\left[\ln\left|\frac{x+1}{n-x}\right| - \ln\frac{1}{n}\right] = kt$$

$$\ln\frac{nx+n}{n-x} = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$x = \frac{n\left[e^{(n+1)kt} - 1\right]}{n + e^{(n+1)kt}}$$

Note: $\lim_{t\to\infty} x = n$

50. (a) $\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x},$

$A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0}$, (Assume $y_0 \neq z_0$.)

$$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$$

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$$

$$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0\left[e^{(z_0 - y_0)kt} - 1\right]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(3) If $y_0 = z_0$, then the original equation is:

$$\int \frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kty_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kty_0 + 1}$$

$$x = y_0 - \frac{y_0}{kty_0 + 1}$$

As $t \to \infty$, $x \to y_0 = x_0$.

51. $$\frac{x}{1 + x^4} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}$$

$$x = (Ax + B)\left(x^2 - \sqrt{2}x + 1\right) + (Cx + D)\left(x^2 + \sqrt{2}x + 1\right)$$

$$= (A + C)x^3 + \left(B + D - \sqrt{2}A + \sqrt{2}C\right)x^2 + \left(A + C - \sqrt{2}B + \sqrt{2}D\right)x + (B + D)$$

$0 = A + C \Rightarrow C = -A$

$0 = B + D - \sqrt{2}A + \sqrt{2}C \qquad -2\sqrt{2}A = 0 \Rightarrow A = 0$ and $C = 0$

$1 = A + C - \sqrt{2}B + \sqrt{2}D \qquad -2\sqrt{2}B = 1 \Rightarrow B = -\dfrac{\sqrt{2}}{4}$ and $D = \dfrac{\sqrt{2}}{4}$

$0 = B + D \Rightarrow D = -B$

So,

$$\int_0^1 \frac{x}{1 + x^4}\,dx = \int_0^1 \left(\frac{-\sqrt{2}/4}{x^2 + \sqrt{2}x + 1} + \frac{\sqrt{2}/4}{x^2 - \sqrt{2}x + 1}\right)dx$$

$$= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x + \left(\sqrt{2}/2\right)\right]^2 + (1/2)} + \frac{1}{\left[x - \left(\sqrt{2}/2\right)\right]^2 + (1/2)}\right]dx$$

$$= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x + \left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x - \left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right)\right]_0^1$$

$$= \frac{1}{2}\left[-\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\right]_0^1$$

$$= \frac{1}{2}\left[\left(-\arctan\left(\sqrt{2} + 1\right) + \arctan\left(\sqrt{2} - 1\right)\right) - \left(-\arctan 1 + \arctan(-1)\right)\right]$$

$$= \frac{1}{2}\left[\arctan\left(\sqrt{2} - 1\right) - \arctan\left(\sqrt{2} + 1\right) + \frac{\pi}{4} + \frac{\pi}{4}\right].$$

Because $\arctan x - \arctan y = \arctan\left[(x - y)/(1 + xy)\right]$, you have:

$$\int_0^1 \frac{x}{1 + x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{\left(\sqrt{2} - 1\right) - \left(\sqrt{2} + 1\right)}{1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} + 1\right)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{\pi}{8}$$

The partial fraction decomposition is:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right)$$

$$= \frac{22}{7} - \pi$$

Note: You can easily verify this calculation with a graphing utility.

Section 8.6 Integration by Tables and Other Integration Techniques

1. By Formula 6: $(a = 5, b = 1)$

$$\int \frac{x^2}{5+x}\,dx = \left[-\frac{x}{2}(10 - x) + 25\ln|5 + x|\right] + C$$

2. By Formula 13: $(a = 4, b = 3)$

$$\int \frac{2}{x^2(4+3x)^2}\,dx = 2\left(\frac{-1}{16}\right)\left[\frac{4+6x}{x(4+3x)} + \frac{6}{4}\ln\left|\frac{x}{4+3x}\right|\right] + C$$

$$= -\frac{(2+3x)}{4x(3x+4)} - \frac{3}{16}\ln\left|\frac{x}{4+3x}\right| + C$$

3. By Formula 44: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

4. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{\sqrt{64-x^4}}{x}\,du = \frac{1}{2}\int \frac{\sqrt{64-x^4}}{x^2}(2x\,dx)$$

$$= \frac{1}{2}\int \frac{\sqrt{64-u^2}}{u}\,du$$

By Formula 39: $(a = 8)$

$$\int \frac{\sqrt{64-u^2}}{u}\,dx = \frac{1}{2}\left[\sqrt{64-u^2} - 8\ln\left|\frac{8+\sqrt{64-u^2}}{u}\right|\right] + C$$

$$= \frac{1}{2}\sqrt{64-x^4} - 4\ln\left|\frac{8+\sqrt{64-x^4}}{x^2}\right| + C$$

5. By Formulas 51 and 49:

$$\begin{aligned}\int \cos^4 3x\,dx &= \frac{1}{3}\int \cos^4 3x\,(3)\,dx\\ &= \frac{1}{3}\left[\frac{\cos^3 3x \sin 3x}{4} + \frac{3}{4}\int \cos^2 3x\,dx\right]\\ &= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{4}\cdot\frac{1}{3}\int \cos^2 3x\,(3)\,dx\\ &= \frac{1}{12}\cos^3 3x \sin 3x + \frac{1}{12}\cdot\frac{1}{2}(3x + \sin 3x \cos 3x) + C\\ &= \frac{1}{24}\left(2\cos^3 3x \sin 3x + 3x + \sin 3x \cos 3x\right) + C\end{aligned}$$

6. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\begin{aligned}\int \frac{\sin^4\sqrt{x}}{\sqrt{x}}\,dx &= 2\int \sin^4 u\,du\\ &= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\int \sin^2 u\,du\right] && \text{(Formula 50, } n = 4\text{)}\\ &= 2\left[-\frac{\sin^3 u \cos u}{4} + \frac{3}{4}\cdot\frac{1}{2}(u - \sin u \cos u)\right] + C && \text{(Formula 48)}\\ &= -\frac{1}{2}\sin^3 u \cos u + \frac{3}{4}u - \frac{3}{4}\sin u \cos u + C\\ &= -\frac{1}{2}\sin^3\sqrt{x}\,\cos\sqrt{x} + \frac{3}{4}\sqrt{x} - \frac{3}{4}\sin\sqrt{x}\cos\sqrt{x} + C\end{aligned}$$

7. By Formula 57: $\displaystyle\int \frac{1}{\sqrt{x}\left(1 - \cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1 - \cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$

$u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$

8. Let $u = 4x$, $du = 4\,dx$.

By Formula 72:

$$\begin{aligned}\int \frac{1}{1 + \cot 4x}\,dx &= \frac{1}{4}\int \frac{1}{1 + \cot 4x}(4\,dx)\\ &= \frac{1}{4}\cdot\frac{1}{2}\left(4x - \ln\left|\sin 4x + \cos 4x\right|\right) + C\\ &= \frac{1}{2}x - \frac{1}{8}\ln\left|\sin 4x + \cos 4x\right| + C\end{aligned}$$

9. By Formula 84:

$$\int \frac{1}{1 + e^{2x}}\,dx = 2x - \frac{1}{2}\ln\left(1 + e^{2x}\right) + C$$

10. By Formula 85: $(a = -4, b = 3)$

$$\begin{aligned}\int e^{-4x}\sin 3x\,dx &= \frac{e^{-4x}}{(-4)^2 + 3^2}(-4\sin 3x - 3\cos 3x) + C\\ &= \frac{e^{-4x}}{25}(-4\sin 3x - 3\cos 3x) + C\end{aligned}$$

11. By Formula 89: $(n = 7)$

$$\int x^7 \ln x\,dx = \frac{x^8}{64}[-1 + 8\ln x] + C = \frac{1}{64}x^8(8\ln x - 1) + C$$

12. By Formulas 90 and 91:
$$\begin{aligned}\int(\ln x)^3\,dx &= x(\ln x)^3 - 3\int(\ln x)^2\,dx\\ &= x(\ln x)^3 - 3x\left[2 - 2\ln x + (\ln x)^2\right] + C\\ &= x\left[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\right] + C\end{aligned}$$

13. (a) Let $u = 3x$, $x = \frac{u}{3}$, $du = 3\,dx$.

$$\int x^2e^{3x}\,dx = \int\left(\frac{u}{3}\right)^2 e^u\frac{1}{3}du = \frac{1}{27}\int u^2e^u\,du$$

By Formulas 83 and 82:

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{1}{27}\left[u^2e^u - 2\int ue^u\,du\right]\\ &= \frac{1}{27}\left[u^2e^u - 2\left((u-1)e^u\right)\right] + C\\ &= \frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C\end{aligned}$$

(b) Integration by parts:

$u = x^2$, $du = 2x\,dx$, $dv = e^{3x}\,dx$, $v = \frac{1}{3}e^{3x}$

$$\int x^2e^{3x}\,dx = x^2\frac{1}{3}e^{3x} - \int\frac{2}{3}xe^{3x}\,dx$$

Parts again: $u = x$, $du = dx$, $dv = e^{3x}$, $v = \frac{1}{3}e^{3x}$

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \int\frac{1}{3}e^{3x}\,dx\right]\\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C\\ &= \frac{1}{27}e^{3x}\left[9x^2 - 6x + 2\right] + C\end{aligned}$$

14. (a) By Formula 89: $(n = 5)$

$$\int x^5\ln x\,dx = \frac{x^6}{36}\ [-1 + 6\ln x] + C$$

(b) Integration by parts:

$u = \ln x$, $du = \frac{1}{x}dx$, $dv = x^5\,dx$, $v = \frac{x^6}{6}$

$$\begin{aligned}\int x^5\ln x\,dx &= \frac{x^6}{6}\ln x - \int\frac{x^6}{6}\cdot\frac{1}{x}\,dx\\ &= \frac{x^6}{6}\ln x - \frac{x^6}{36} + C\end{aligned}$$

15. (a) By Formula 12: $(a = b = 1, u = x)$

$$\begin{aligned}\int\frac{1}{x^2(x+1)}\,dx &= \frac{-1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1+x}\right|\right) + C\\ &= \frac{-1}{x} - \ln\left|\frac{x}{1+x}\right| + C\\ &= \frac{-1}{x} + \ln\left|\frac{x+1}{x}\right| + C\end{aligned}$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$
$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$x = 0$: $1 = B$

$x = -1$: $1 = C$

$x = 1$: $1 = 2A + 2 + 1 \Rightarrow A = -1$

$$\begin{aligned}\int\frac{1}{x^2(x+1)}\,dx &= \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right]dx\\ &= -\ln|x| - \frac{1}{x} + \ln|x+1| + C\\ &= -\frac{1}{x} - \ln\left|\frac{x}{x+1}\right| + C\end{aligned}$$

16. (a) By Formula 24: $(a = 6)$

$$\int\frac{1}{x^2 - 36}\,dx = \frac{1}{12}\ln\left|\frac{x-6}{x+6}\right| + C$$

(b) Partial Fractions:

$$\frac{1}{x^2 - 36} = \frac{A}{x-6} + \frac{B}{x+6}$$
$$1 = A(x+6) + B(x-6)$$

When $x = -6$, $1 = -12B \Rightarrow B = -\frac{1}{12}$.

When $x = 6$, $1 = 12A \Rightarrow A = \frac{1}{12}$.

$$\begin{aligned}\int\frac{1}{x^2 - 36}\,dx &= \int\frac{1/12}{x-6}\,dx + \int\frac{-1/12}{x+6}\,dx\\ &= \frac{1}{12}\ln|x-6| - \frac{1}{12}\ln|x+6| + C\\ &= \frac{1}{12}\ln\left|\frac{x-6}{x+6}\right| + C\end{aligned}$$

17. By Formula 80:

$$\int x \operatorname{arccsc}(x^2+1)\,dx = \frac{1}{2}\int \operatorname{arccsc}(x^2+1)(2x)\,dx$$
$$= \frac{1}{2}\left[(x^2+1)\operatorname{arccsc}(x^2+1) + \ln\left|x^2+1+\sqrt{(x^2+1)^2-1}\right|\right] + C$$
$$= \frac{1}{2}(x^2+1)\operatorname{arccsc}(x^2+1) + \frac{1}{2}\ln\left(x^2+1+\sqrt{x^4+2x^2}\right) + C$$

18. By Formula 75: $u = 4x$

$$\int \arcsin 4x\,dx = \frac{1}{4}\int \arcsin 4x(4\,dx)$$
$$= \frac{1}{4}\left[4x\arcsin 4x + \sqrt{1-(4x)^2}\right] + C$$
$$= x\arcsin 4x + \frac{1}{4}\sqrt{1-16x^2} + C$$

19. By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{\sqrt{x^2-4}}{4x} + C$

20. By Formula 14: $(a = 8, b = 4, c = 1, b^2 < 4ac)$

$$\int \frac{1}{x^2+4x+8}\,dx = \frac{2}{\sqrt{16}}\arctan\frac{2x+4}{\sqrt{16}} + C$$
$$= \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C$$

21. By Formula 4: $(a = 2, b = -5)$

$$\int \frac{4x}{(2-5x)^2}\,dx = 4\left[\frac{1}{25}\left(\frac{2}{2-5x} + \ln|2-5x|\right)\right] + C$$
$$= \frac{4}{25}\left(\frac{2}{2-5x} + \ln|2-5x|\right) + C$$

22. By Formula 56: $u = \theta^4$, $du = 4\theta^3\,d\theta$

$$\int \frac{\theta^3}{1+\sin\theta^4}\,d\theta = \frac{1}{4}\int \frac{1}{1+\sin\theta^4}4\theta^3\,d\theta$$
$$= \frac{1}{4}\left(\tan\theta^4 - \sec\theta^4\right) + C$$

23. By Formula 76:

$$\int e^x \arccos e^x\,dx = e^x\arccos e^x - \sqrt{1-e^{2x}} + C$$
$u = e^x$, $du = e^x\,dx$

24. By Formula 71:

$$\int \frac{e^x}{1-\tan e^x}\,dx = \frac{1}{2}\left(e^x - \ln\left|\cos e^x - \sin e^x\right|\right) + C$$
$u = e^x$, $du = e^x\,dx$

25. By Formula 73:

$$\int \frac{x}{1-\sec x^2}\,dx = \frac{1}{2}\int \frac{2x}{1-\sec x^2}\,dx$$
$$= \frac{1}{2}\left(x^2 + \cot x^2 + \csc x^2\right) + C$$

26. By Formula 23: $\displaystyle\int \frac{1}{t\left[1+(\ln t)^2\right]}\,dt = \int \frac{1}{1+(\ln t)^2}\left(\frac{1}{t}\right)dt = \arctan(\ln t) + C$

$u = \ln t$, $du = \frac{1}{t}\,dt$

27. By Formula 14: $\displaystyle\int \frac{\cos\theta}{3+2\sin\theta+\sin^2\theta}\,d\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{1+\sin\theta}{\sqrt{2}}\right) + C \quad (b^2 = 4 < 12 = 4ac)$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

28. By Formula 27: $\displaystyle\int x^2\sqrt{2+(3x)^2}\,dx = \frac{1}{27}\int (3x)^2\sqrt{\left(\sqrt{2}\right)^2+(3x)^2}\,3\,dx$

$$= \frac{1}{8(27)}\left[3x(18x^2+2)\sqrt{2+9x^2} - 4\ln\left|3x+\sqrt{2+9x^2}\right|\right] + C$$

29. By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{2+9x^2}}\,dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2+(3x)^2}}\,dx = -\frac{3\sqrt{2+9x^2}}{6x} + C = -\frac{\sqrt{2+9x^2}}{2x} + C$

30. By Formula 77: $\int \sqrt{x} \arctan\left(x^{3/2}\right) dx = \frac{2}{3} \int \arctan\left(x^{3/2}\right)\left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3}\left[x^{3/2} \arctan\left(x^{3/2}\right) - \ln \sqrt{1 + x^3}\right] + C$

31. By Formula 3: $\int \frac{\ln x}{x(3 + 2 \ln x)} dx = \frac{1}{4}\left(2 \ln|x| - 3 \ln|3 + 2 \ln|x||\right) + C$

$u = \ln x, du = \frac{1}{x} dx$

32. By Formula 45: $\int \frac{e^x}{\left(1 - e^{2x}\right)^{3/2}} dx = \frac{e^x}{\sqrt{1 - e^{2x}}} + C$

$u = e^x, du = e^x dx$

33. By Formulas 1, 23, and 35: $\int \frac{x}{\left(x^2 - 6x + 10\right)^2} dx = \frac{1}{2}\int \frac{2x - 6 + 6}{\left(x^2 - 6x + 10\right)^2} dx$

$$= \frac{1}{2}\int \left(x^2 - 6x + 10\right)^{-2}(2x - 6)\, dx + 3\int \frac{1}{\left[(x - 3)^2 + 1\right]^2} dx$$

$$= -\frac{1}{2\left(x^2 - 6x + 10\right)} + \frac{3}{2}\left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3)\right] + C$$

$$= \frac{3x - 10}{2\left(x^2 - 6x + 10\right)} + \frac{3}{2} \arctan(x - 3) + C$$

34. By Formula 41:

$$\int \sqrt{\frac{5 - x}{5 + x}} dx = \int \frac{\sqrt{5 - x}}{\sqrt{5 + x}} \cdot \frac{\sqrt{5 - x}}{\sqrt{5 - x}} dx$$

$$= \int \frac{5 - x}{\sqrt{25 - x^2}} dx$$

$$= \int \frac{5\, dx}{\sqrt{25 - x^2}} - \int \frac{x}{\sqrt{25 - x^2}} dx$$

$$= 5 \arcsin\left(\frac{x}{5}\right) + \sqrt{25 - x^2} + C$$

35. By Formula 31: $\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}} dx = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2 - 3\right)^2 - 4}} dx = \frac{1}{2} \ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$

$u = x^2 - 3, du = 2x\, dx$

36. By Formula 31: $\int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx = \ln\left|\sin x + \sqrt{\sin^2 x + 1}\right| + C$

$u = \sin x, du = \cos x\, dx$

37. By Formula 8:

$$\int \frac{e^{3x}}{\left(1 + e^x\right)^3} dx = \int \frac{\left(e^x\right)^2}{\left(1 + e^x\right)^3}\left(e^x\right) dx$$

$$= \frac{2}{1 + e^x} - \frac{1}{2\left(1 + e^x\right)^2} + \ln\left|1 + e^x\right| + C$$

$u = e^x, du = e^x dx$

38. By Formulas 64 and 68:

$$\int \cot^4 \theta \, d\theta = -\frac{\cot^3 \theta}{3} - \int \cot^2 \theta \, d\theta$$
$$= -\frac{\cot^3 \theta}{3} + \theta + \cot \theta + C$$

39. By Formula 81:

$$\int_0^1 xe^{x^2}\,dx = \left[\tfrac{1}{2}e^{x^2}\right]_0^1 = \tfrac{1}{2}(e-1) \approx 0.8591$$

40. By Formula 21: $(a = 3, b = 2)$

$$\int_0^4 \frac{x}{\sqrt{3+2x}}\,dx = \left[\frac{-2(6-2x)}{12}\sqrt{3+2x}\right]_0^4$$
$$= \left[-\frac{1}{6}(6-2x)\sqrt{3+2x}\right]_0^4$$
$$= -\frac{1}{6}(-2)\sqrt{11} + \frac{1}{6}(6)\sqrt{3}$$
$$= \frac{\sqrt{11}}{3} + \sqrt{3}$$

41. By Formula 89: $(n = 4)$

$$\int_1^2 x^4 \ln x\,dx = \left[\frac{x^5}{25}(-1 + 5\ln x)\right]_1^2$$
$$= \frac{32}{25}[-1 + 5\ln 2] - \frac{1}{25}[-1 + 0]$$
$$= -\frac{31}{25} + \frac{32}{5}\ln 2 \approx 3.1961$$

42. By Formula 52: $u = 2x$, $du = 2dx$

$$\int_0^{\pi/2} x \sin 2x\,dx = \frac{1}{4}\int_0^{\pi/2}(2x)\sin 2x\,(2dx)$$
$$= \frac{1}{4}[\sin 2x - 2x\cos 2x]_0^{\pi/2}$$
$$= \frac{1}{4}[0 - \pi(-1)]$$
$$= \frac{\pi}{4}$$

43. By Formula 23, and letting $u = \sin x$:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \left[\arctan(\sin x)\right]_{-\pi/2}^{\pi/2}$$
$$= \arctan(1) - \arctan(-1) = \frac{\pi}{2}$$

44. By Formula 7: $(a = 5, b = 2)$

$$\int_0^5 \frac{x^2}{(5+2x)^2}\,dx = \frac{1}{8}\left[2x - \frac{25}{5+2x} - 10\ln|5+2x|\right]_0^5$$
$$= \frac{1}{8}\left[\left(10 - \frac{25}{15} - 10\ln 15\right) - (-5 - 10\ln 5)\right]$$
$$= \frac{5}{3} - \frac{1}{8}(10)\ln\left(\frac{15}{5}\right)$$
$$= \frac{5}{3} - \frac{5}{4}\ln 3$$

45. By Formulas 54 and 55:

$$\int t^3 \cos t\,dt = t^3 \sin t - 3\int t^2 \sin t\,dt$$
$$= t^3 \sin t - 3\left(-t^2 \cos t + 2\int t\cos t\,dt\right)$$
$$= t^3 \sin t + 3t^2 \cos t - 6\left(t \sin t - \int \sin t\,dt\right)$$
$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6\cos t + C$$

So,

$$\int_0^{\pi/2} t^3 \cos t\,dt = \left[t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6\cos t\right]_0^{\pi/2}$$
$$= \left(\frac{\pi^3}{8} - 3\pi\right) + 6 = \frac{\pi^3}{8} + 6 - 3\pi \approx 0.4510.$$

46. By Formula 26: $(a = 4)$

$$\begin{aligned}\int_0^3 \sqrt{x^2+16}\,dx &= \frac{1}{2}\left[x\sqrt{x^2+16} + 16\ln\left|x+\sqrt{x^2+16}\right|\right]_0^3 \\ &= \frac{1}{2}\left[\left(3(5) + 16\ln|3+5|\right) - \left(16\ln 4\right)\right] \\ &= \frac{15}{2} + 8\ln 8 - 8\ln 4 \\ &= \frac{15}{2} + 8\ln 2\end{aligned}$$

47. $$\frac{u^2}{(a+bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a+bu)^2} = \frac{1}{b^2} + \frac{A}{a+bu} + \frac{B}{(a+bu)^2}$$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a+bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms you have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations you have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\begin{aligned}\int \frac{u^2}{(a+bu)^2}\,du &= \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a+bu}b\,du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a+bu)^2}b\,du = \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a+bu| - \frac{a^2}{b^3}\left(\frac{1}{a+bu}\right) + C \\ &= \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C\end{aligned}$$

48. Integration by parts: $w = u^n$, $dw = nu^{n-1}\,du$, $dv = \dfrac{du}{\sqrt{a+bu}}$, $v = \dfrac{2}{b}\sqrt{a+bu}$

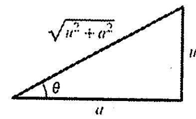
$$\begin{aligned}\int \frac{u^n}{\sqrt{a+bu}}\,du &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\,du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\cdot\frac{\sqrt{a+bu}}{\sqrt{a+bu}}\,du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int \frac{au^{n-1} + bu^n}{\sqrt{a+bu}}\,du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du - 2n\int \frac{u^n}{\sqrt{a+bu}}\,du\end{aligned}$$

Therefore, $(2n+1)\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right]$ and

$$\int \frac{u^n}{\sqrt{a+bu}} = \frac{2}{(2n+1)b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right].$$

49. When you have $u^2 + a^2$:

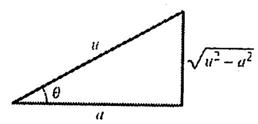
$$u = a\tan\theta$$
$$du = a\sec^2\theta\,d\theta$$
$$u^2 + a^2 = a^2\sec^2\theta$$



$$\int\frac{1}{\left(u^2+a^2\right)^{3/2}}\,du = \int\frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C = \frac{u}{a^2\sqrt{u^2+a^2}} + C$$

When you have $u^2 - a^2$:

$$u = a\sec\theta$$
$$du = a\sec\theta\tan\theta\,d\theta$$
$$u^2 - a^2 = a^2\tan^2\theta$$



$$\int\frac{1}{\left(u^2-a^2\right)^{3/2}}\,du = \int\frac{a\sec\theta\tan\theta\,d\theta}{a^3\tan^3\theta} = \frac{1}{a^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{a^2}\int\csc\theta\cot\theta\,d\theta = -\frac{1}{a^2}\csc\theta + C = \frac{-u}{a^2\sqrt{u^2-a^2}} + C$$

50. $\int u^n(\cos u)\,du = u^n\sin u - n\int u^{n-1}(\sin u)\,du$

$w = u^n,\ dv = \cos u\,du,\ dw = nu^{n-1}\,du,\ v = \sin u$

51. $\int(\arctan u)\,du = u\arctan u - \frac{1}{2}\int\frac{2u}{1+u^2}\,du$

$$= u\arctan u - \frac{1}{2}\ln\left(1+u^2\right) + C$$
$$= u\arctan u - \ln\sqrt{1+u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \dfrac{du}{1+u^2},\ v = u$

52. $\int(\ln u)^n\,du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\,du$

$$= u(\ln u)^n - n\int(\ln u)^{n-1}\,du$$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\dfrac{1}{u}\right)du,$

$v = u$

53. $\int\frac{1}{2-3\sin\theta}\,d\theta = \int\left[\frac{\frac{2\,du}{1+u^2}}{2-3\left(\frac{2u}{1+u^2}\right)}\right],\ u = \tan\frac{\theta}{2}$

$$= \int\frac{2}{2\left(1+u^2\right)-6u}\,du$$
$$= \int\frac{1}{u^2-3u+1}\,du$$
$$= \int\frac{1}{\left(u-\frac{3}{2}\right)^2-\frac{5}{4}}\,du$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u-\frac{3}{2}\right)-\frac{\sqrt{5}}{2}}{\left(u-\frac{3}{2}\right)+\frac{\sqrt{5}}{2}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2u-3-\sqrt{5}}{2u-3+\sqrt{5}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\frac{\theta}{2}\right)-3-\sqrt{5}}{2\tan\left(\frac{\theta}{2}\right)-3+\sqrt{5}}\right| + C$$

54. $\int\frac{\sin\theta}{1+\cos^2\theta}\,d\theta = -\int\frac{-\sin\theta}{1+(\cos\theta)^2}\,d\theta$

$$= -\arctan(\cos\theta) + C$$

55. $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sin\theta + \cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2\,du}{1+u^2}}{1 + \dfrac{2u}{1+u^2} + \dfrac{1-u^2}{1+u^2}}\right]$

$\displaystyle = \int_0^1 \frac{1}{1+u}\,du$

$\displaystyle = \Big[\ln|1+u|\Big]_0^1$

$= \ln 2$

$u = \tan\dfrac{\theta}{2}$

56. $\displaystyle\int_0^{\pi/2} \frac{1}{3 - 2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2u}{1+u^2}}{3 - \dfrac{2(1-u^2)}{1+u^2}}\right]$

$\displaystyle = 2\int_0^1 \frac{1}{5u^2+1}\,du$

$\displaystyle = \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$

$\displaystyle = \frac{2}{\sqrt{5}}\arctan\sqrt{5}$

$u = \tan\dfrac{\theta}{2}$

57. $\displaystyle\int \frac{\sin\theta}{3 - 2\cos\theta}\,d\theta = \frac{1}{2}\int \frac{2\sin\theta}{3 - 2\cos\theta}\,d\theta$

$\displaystyle = \frac{1}{2}\ln|u| + C$

$\displaystyle = \frac{1}{2}\ln(3 - 2\cos\theta) + C$

$u = 3 - 2\cos\theta,\ du = 2\sin\theta\,d\theta$

58. $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$

$\displaystyle = \int \frac{\cos\theta - \cos^2\theta}{\sin^2\theta}\,d\theta$

$\displaystyle = \int\left(\csc\theta\cot\theta - \cot^2\theta\right)d\theta$

$\displaystyle = \int\left(\csc\theta\cot\theta - \left(\csc^2\theta - 1\right)\right)d\theta$

$= -\csc\theta + \cot\theta + \theta + C$

59. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \sin\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$

$= -2\cos\sqrt{\theta} + C$

$u = \sqrt{\theta},\ du = \dfrac{1}{2\sqrt{\theta}}\,d\theta$

60. $\displaystyle\int \frac{4}{\csc\theta - \cot\theta}\,d\theta = \int \frac{4}{\left(\dfrac{1}{\sin\theta}\right) - \left(\dfrac{\cos\theta}{\sin\theta}\right)}\,d\theta$

$\displaystyle = 4\int \frac{\sin\theta}{1 - \cos\theta}\,d\theta$

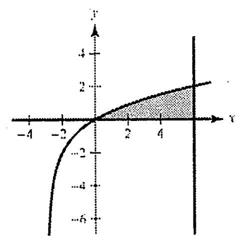
$= 4\ln|1 - \cos\theta| + C$

$u = 1 - \cos\theta,\ du = \sin\theta\,d\theta$

61. By Formula 21: $(a = 3, b = 1)$

$\displaystyle A = \int_0^6 \frac{x}{\sqrt{x+3}}\,dx = \left[\frac{-2(6-x)}{3}\sqrt{x+3}\right]_0^6$

$= 4\sqrt{3} \approx 6.928$ square units



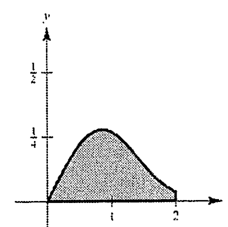
62. $\displaystyle A = \int_0^2 \frac{x}{1+e^{x^2}}\,dx$

$\displaystyle = \frac{1}{2}\int_0^2 \frac{2x\,dx}{1+e^{x^2}}$

$\displaystyle = \frac{1}{2}\left[x^2 - \ln\left(1 + e^{x^2}\right)\right]_0^2$

$\displaystyle = \frac{1}{2}\left[4 - \ln\left(1 + e^4\right)\right] + \frac{1}{2}\ln 2$

≈ 0.337 square units



63. (a) $n = 1$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x\,dx,\ v = \frac{x^2}{2}$

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\right)\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$n = 2$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x^2\,dx,\ v = \frac{x^3}{3}$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int\left(\frac{x^3}{3}\right)\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$n = 3$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x^3\,dx,\ v = \frac{x^4}{4}$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int\left(\frac{x^4}{4}\right)\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

(b) $\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$

64. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formulas 50, 54.

65. (a) Arctangent Formula, Formula 23,

$\int \frac{1}{u^2 + 1}\,du,\ u = e^x$

(b) Log Rule: $\int \frac{1}{u}\,du,\ u = e^x + 1$

(c) Substitution: $u = x^2,\ du = 2x\,dx$, then Formula 81

(d) Integration by parts

(e) Cannot be integrated.

(f) Formula 16 with $u = e^{2x}$

66. (a) The slope of f at $x = -1$ is approximately $0.5\ (f' > 0 \text{ at } x = -1)$.

(b) $f' > 0$ on $(-\infty, 0)$, so f is increasing on $(-\infty, 0)$.

$f' < 0$ on $(0, \infty)$, so f is decreasing on $(0, \infty)$.

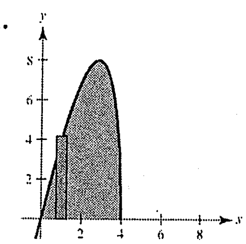
67. False. You might need to convert your integral using substitution or algebra.

68. True

69.
$$\begin{aligned} W &= \int_0^5 2000xe^{-x}\,dx \\ &= -2000\int_0^5 -xe^{-x}\,dx \\ &= 2000\int_0^5 (-x)e^{-x}(-1)\,dx \\ &= 2000\Big[(-x)e^{-x} - e^{-x}\Big]_0^5 \\ &= 2000\left(-\frac{6}{e^5} + 1\right) \\ &\approx 1919.145 \text{ ft-lb} \end{aligned}$$

70.
$$\begin{aligned} W &= \int_0^5 \frac{500x}{\sqrt{26 - x^2}}\,dx \\ &= -250\int_0^5 \left(26 - x^2\right)^{-1/2}(-2x)\,dx \\ &= \Big[-500\sqrt{26 - x^2}\Big]_0^5 \\ &= 500\left(\sqrt{26} - 1\right) \\ &\approx 2049.51 \text{ ft-lb} \end{aligned}$$

71.



$$\begin{aligned} V &= 2\pi\int_0^4 x\left(x\sqrt{16 - x^2}\right)dx \\ &= 2\pi\int_0^4 x^2\sqrt{16 - x^2}\,dx \end{aligned}$$

By Formula 38: $(a = 4)$

$$\begin{aligned} V &= 2\pi\left[\frac{1}{8}\left(x\left(2x^2 - 16\right)\sqrt{16 - x^2} + 256\arcsin\left(\frac{x}{4}\right)\right)\right]_0^4 \\ &= 2\pi\left[32\left(\frac{\pi}{2}\right)\right] = 32\pi^2 \end{aligned}$$

72. (a) $V = 20(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$

$= \left[80 \ln\left|y + \sqrt{1+y^2}\right|\right]_0^3$

$= 80 \ln\left(3 + \sqrt{10}\right)$

$\approx 145.5 \text{ ft}^3$

$W = 148\left(80 \ln\left(3 + \sqrt{10}\right)\right)$

$= 11{,}840 \ln\left(3 + \sqrt{10}\right)$

$\approx 21{,}530.4 \text{ lb}$

(b) By symmetry, $\bar{x} = 0$.

$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy$

$= \left[4\rho \ln\left|y + \sqrt{1+y^2}\right|\right]_0^3$

$= 4\rho \ln\left(3 + \sqrt{10}\right)$

$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1+y^2}}\,dy$

$= \left[4\rho\sqrt{1+y^2}\right]_0^3$

$= 4\rho\left(\sqrt{10} - 1\right)$

$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10}-1\right)}{4\rho \ln\left(3+\sqrt{10}\right)} \approx 1.19$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

73. $\frac{1}{2-0}\int_0^2 \frac{5000}{1+e^{4.8-1.9t}}\,dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\,dt}{1+e^{4.8-1.9t}}$

$= -\frac{2500}{1.9}\left[(4.8 - 1.9t) - \ln\left(1 + e^{4.8-1.9t}\right)\right]_0^2$

$= -\frac{2500}{1.9}\left[\left(1 - \ln(1+e)\right) - \left(4.8 - \ln\left(1 + e^{4.8}\right)\right)\right]$

$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1+e}{1+e^{4.8}}\right)\right] \approx 401.4$

74. Let $I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.

For $x = \frac{\pi}{2} - u$, $dx = -du$, and

$$I = \int_{\pi/2}^0 \frac{-du}{1 + \left(\tan(\pi/2 - u)\right)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{du}{1 + (\cot u)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}} + 1}\,du.$$

$$2I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{(\tan x)^{\sqrt{2}} + 1}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

So, $I = \frac{\pi}{4}$.