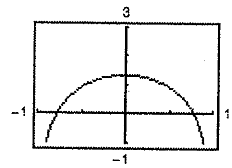
Section 8.7 Indeterminate Forms and L'Hôpital's Rule

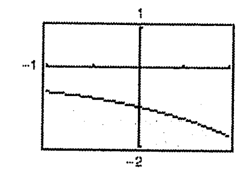
1. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 3x} \approx 1.3333 \quad \left(\text{exact: } \dfrac{4}{3}\right)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177



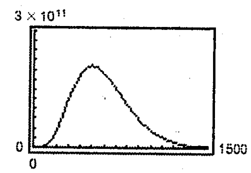
2. $\lim_{x\to 0} \dfrac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



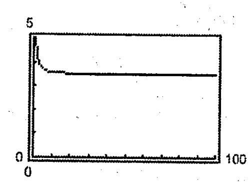
3. $\lim_{x\to\infty} x^5 e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,484	3.7×10^9	4.5×10^{10}	0	0



4. $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \quad \left(\text{exact: } \dfrac{6}{\sqrt{3}}\right)$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



5. (a) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2 - 16} = \lim_{x\to 4} \dfrac{3(x-4)}{(x-4)(x+4)} = \lim_{x\to 4} \dfrac{3}{x+4} = \dfrac{3}{8}$

(b) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2 - 16} = \lim_{x\to 4} \dfrac{d/dx[3(x-4)]}{d/dx[x^2 - 16]} = \lim_{x\to 4} \dfrac{3}{2x} = \dfrac{3}{8}$

6. (a) $\lim_{x\to -4} \dfrac{2x^2 + 13x + 20}{x + 4} = \lim_{x\to -4} \dfrac{(x+4)(2x+5)}{x+4} = \lim_{x\to -4}(2x + 5) = -8 + 5 = -3$

(b) $\lim_{x\to -4} \dfrac{2x^2 + 13x + 20}{x + 4} = \lim_{x\to -4} \dfrac{d/dx[2x^2 + 13x + 20]}{d/dx[x + 4]} = \lim_{x\to -4} \dfrac{4x + 13}{1} = -3$

7. (a) $\lim_{x\to 6} \dfrac{\sqrt{x+10} - 4}{x - 6} = \lim_{x\to 6} \dfrac{\sqrt{x+10} - 4}{x - 6} \cdot \dfrac{\sqrt{x+10} + 4}{\sqrt{x+10} + 4} = \lim_{x\to 6} \dfrac{(x+10) - 16}{(x-6)\left(\sqrt{x+10} + 4\right)} = \lim_{x\to 6} \dfrac{1}{\sqrt{x+10} + 4} = \dfrac{1}{8}$

(b) $\lim_{x\to 6} \dfrac{\sqrt{x+10} - 4}{x - 6} = \lim_{x\to 6} \dfrac{d/dx\left[\sqrt{x+10} - 4\right]}{d/dx[x - 6]} = \lim_{x\to 6} \dfrac{\frac{1}{2}(x+10)^{-1/2}}{1} = 1/8$

8. (a) $\lim_{x\to 0} \frac{\sin 6x}{4x} = \lim_{x\to 0}\left(\frac{3}{2}\cdot\frac{\sin 6x}{6x}\right) = \frac{3}{2}(1) = \frac{3}{2}$

(b) $\lim_{x\to 0} \frac{\sin 6x}{4x} = \lim_{x\to 0} \frac{d/dx[\sin 6x]}{d/dx[4x]} = \lim_{x\to 0} \frac{6\cos 6x}{4} = \frac{3}{2}$

9. (a) $\lim_{x\to\infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x\to\infty} \frac{5 - (3/x) + (1/x^2)}{3 - (5/x^2)} = \frac{5}{3}$

(b) $\lim_{x\to\infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x\to\infty} \frac{(d/dx)[5x^2 - 3x + 1]}{(d/dx)[3x^2 - 5]} = \lim_{x\to\infty} \frac{10x - 3}{6x} = \lim_{x\to\infty} \frac{(d/dx)[10x - 3]}{(d/dx)[6x]} = \lim_{x\to\infty} \frac{10}{6} = \frac{5}{3}$

10. (a) $\lim_{x\to\infty} \frac{4x - 3}{5x^2 + 1} = \lim_{x\to\infty} \frac{4/x - 3/x^2}{5 + 1/x^2} = 0$

(b) $\lim_{x\to\infty} \frac{4x - 3}{5x^2 + 1} = \lim_{x\to\infty} \frac{(d/dx)[4x - 3]}{(d/dx)[5x^2 + 1]} = \lim_{x\to\infty} \frac{4}{10x} = 0$

11. $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x\to 3} \frac{2x - 2}{1} = 4$

12. $\lim_{x\to -2} \frac{x^2 - 3x - 10}{x + 2} = \lim_{x\to -2} \frac{2x - 3}{1} = -7$

13. $\lim_{x\to 0} \frac{\sqrt{25 - x^2} - 5}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 0} \frac{-x}{\sqrt{25 - x^2}} = 0$

14. $\lim_{x\to 5^-} \frac{\sqrt{25 - x^2}}{x - 5} = \lim_{x\to 5^-} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 5^-} \frac{-x}{\sqrt{25 - x^2}} = -\infty$

15. $\lim_{x\to 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x\to 0^+} \frac{e^x - 1}{3x^2} = \lim_{x\to 0^+} \frac{e^x}{6x} = \infty$

16. $\lim_{x\to 1} \frac{\ln x^3}{x^2 - 1} = \lim_{x\to 1} \frac{3\ln x}{x^2 - 1} = \lim_{x\to 1} \frac{3/x}{2x} = \frac{3}{2}$

17. $\lim_{x\to 1} \frac{x^{11} - 1}{x^4 - 1} = \lim_{x\to 1} \frac{11x^{10}}{4x^3} = \frac{11}{4}$

18. $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

19. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$

20. $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

21. $\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{1/\sqrt{1 - x^2}}{1} = 1$

22. $\lim_{x\to 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x\to 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$

23. $\lim_{x\to\infty} \frac{5x^2 + 3x - 1}{4x^2 + 5} = \lim_{x\to\infty} \frac{10x + 3}{8x} = \lim_{x\to\infty} \frac{10}{8} = \frac{5}{4}$

24. $\lim_{x\to\infty} \frac{5x + 3}{x^3 - 6x + 2} = \lim_{x\to\infty} \frac{5}{3x^2 - 6} = 0$

25. $\lim_{x\to\infty} \frac{x^2 + 4x + 7}{x - 6} = \lim_{x\to\infty} \frac{2x + 4}{1} = \infty$

26. $\lim_{x\to\infty} \frac{x^3}{x + 1} = \lim_{x\to\infty} \frac{3x^2}{1} = \infty$

27. $\lim_{x\to\infty} \frac{x^3}{e^{x/2}} = \lim_{x\to\infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x\to\infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x\to\infty} \frac{6}{(1/8)e^{x/2}} = 0$

28. $\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$

$= \lim_{x\to\infty} \frac{6x}{(4x^2 + 2)e^{x^2}}$

$= \lim_{x\to\infty} \frac{6}{4x(2x^2 + 3)e^{x^2}} = 0$

29. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$

Note: L'Hôpital's Rule does not work on this limit. See Exercise 83.

30. $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{1+(1/x)^2}} = \infty$

31. $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by Squeeze Theorem

$\displaystyle\left(\frac{\cos x}{x} \le \frac{1}{x}, \text{ for } x > 0\right)$

32. $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x-\pi} = 0$

Note: Use the Squeeze Theorem for $x > \pi$.

$$-\frac{1}{x-\pi} \le \frac{\sin x}{x-\pi} \le \frac{1}{x-\pi}$$

33. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

34. $\displaystyle\lim_{x\to\infty} \frac{\ln x^4}{x^3} = \lim_{x\to\infty} \frac{4\ln x}{x^3} = \lim_{x\to\infty} \frac{4/x}{3x^2} = \lim_{x\to\infty} \frac{4}{3x^3} = 0$

35.
$$\begin{aligned}\lim_{x\to\infty} \frac{e^x}{x^4} &= \lim_{x\to\infty} \frac{e^x}{4x^3} \\ &= \lim_{x\to\infty} \frac{e^x}{12x^2} \\ &= \lim_{x\to\infty} \frac{e^x}{24x} \\ &= \lim_{x\to\infty} \frac{e^x}{24} = \infty\end{aligned}$$

36. $\displaystyle\lim_{x\to\infty} \frac{e^{x/2}}{x} = \lim_{x\to\infty} \frac{(1/2)e^{x/2}}{1} = \infty$

37. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\tan 9x} = \lim_{x\to 0} \frac{5\cos 5x}{9\sec^2 9x} = \frac{5}{9}$

38. $\displaystyle\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$

39. $\displaystyle\lim_{x\to 0} \frac{\arctan x}{\sin x} = \lim_{x\to 0} \frac{1/(1+x^2)}{\cos x} = 1$

40. $\displaystyle\lim_{x\to 0} \frac{x}{\arctan 2x} = \lim_{x\to 0} \frac{1}{2/(1+4x^2)} = 1/2$

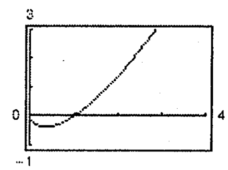
41.
$$\begin{aligned}\lim_{x\to\infty} \frac{\int_1^x \ln\left(e^{4t-1}\right) dt}{x} &= \lim_{x\to\infty} \frac{\int_1^x (4t-1)\, dt}{x} \\ &= \lim_{x\to\infty} \frac{4x-1}{1} = \infty\end{aligned}$$

42. $\displaystyle\lim_{x\to 1^+} \frac{\int_1^x \cos\theta\, d\theta}{x-1} = \lim_{x\to 1^+} \frac{\cos x}{1} = \cos(1)$

43. (a) $\displaystyle\lim_{x\to\infty} x \ln x$, not indeterminate

(b) $\displaystyle\lim_{x\to\infty} x \ln x = (\infty)(\infty) = \infty$

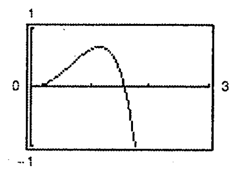
(c)



44. (a) $\displaystyle\lim_{x\to 0^+} x^3 \cot x = (0)(\infty)$

(b) $\displaystyle\lim_{x\to 0^+} x^3 \cot x = \lim_{x\to 0^+} \frac{x^3}{\tan x} = \lim_{x\to 0^+} \frac{3x^2}{\sec^2 x} = 0$

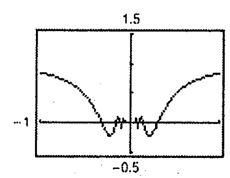
(c)



45. (a) $\displaystyle\lim_{x\to\infty} \left(x \sin\frac{1}{x}\right) = (\infty)(0)$

(b)
$$\begin{aligned}\lim_{x\to\infty} x \sin\frac{1}{x} &= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x\to\infty} \frac{\left(-1/x^2\right)\cos(1/x)}{-1/x^2} \\ &= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1\end{aligned}$$

(c)



46. (a) $\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}$

$$= \lim_{x\to\infty}\frac{-\left(1/x^2\right)\sec^2(1/x)}{-\left(1/x^2\right)}$$

$$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1$$

(c)

47. (a) $\lim_{x\to0^+} x^{1/x} = 0^\infty = 0$, not indeterminate

(See Exercise 108).

(b) Let $y = x^{1/x}$

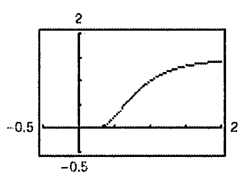
$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x.$$

Because $x \to 0^+, \frac{1}{x}\ln x \to (\infty)(-\infty) = -\infty$. So,

$\ln y \to -\infty \Rightarrow y \to 0^+$.

Therefore, $\lim_{x\to0^+} x^{1/x} = 0$.

(c)



48. (a) $\lim_{x\to0^+}\left(e^x + x\right)^{2/x} = 1^\infty$

(b) Let $y = \lim_{x\to0^+}\left(e^x + x\right)^{2/x}$.

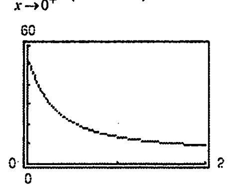
$$\ln y = \lim_{x\to0^+}\frac{2\ln\left(e^x + x\right)}{x}$$

$$= \lim_{x\to0^+}\frac{2\left(e^x + 1\right)/\left(e^x + x\right)}{1} = 4$$

So, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$. Therefore,

$\lim_{x\to0^+}\left(e^x + x\right)^{2/x} = e^4$.

(c)



49. (a) $\lim_{x\to\infty} x^{1/x} = \infty^0$

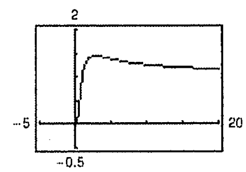
(b) Let $y = \lim_{x\to\infty} x^{1/x}$.

$$\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$\lim_{x\to\infty} x^{1/x} = 1$.

(c)



50. (a) $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

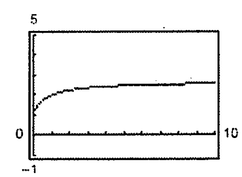
$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1 + \frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left[1 + (1/x)\right]}{1/x}$$

$$= \lim_{x\to\infty}\frac{\left[\frac{\left(-1/x^2\right)}{1 + (1/x)}\right]}{\left(-1/x^2\right)} = \lim_{x\to\infty}\frac{1}{1 + (1/x)} = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$. Therefore,

$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$.

(c)



51. (a) $\lim_{x \to 0^+} (1 + x)^{1/x} = 1^\infty$

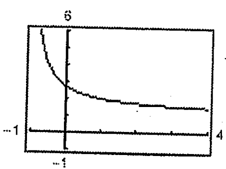
(b) Let $y = \lim_{x \to 0^+} (1 + x)^{1/x}$.

$$\ln y = \lim_{x \to 0^+} \frac{\ln(1 + x)}{x}$$
$$= \lim_{x \to 0^+} \left(\frac{1/(1 + x)}{1} \right) = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

(c)



52. (a) $\lim_{x \to \infty} (1 + x)^{1/x} = \infty^0$

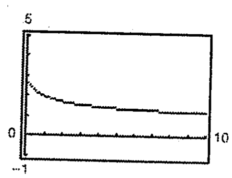
(b) Let $y = \lim_{x \to \infty} (1 + x)^{1/x}$.

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + x)}{x} = \lim_{x \to \infty} \left(\frac{1/(1 + x)}{1} \right) = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x \to \infty} (1 + x)^{1/x} = 1$.

(c)



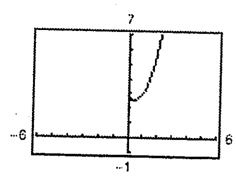
53. (a) $\lim_{x \to 0^+} \left[3(x)^{x/2} \right] = 0^0$

(b) Let $y = \lim_{x \to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x \to 0^+} \left[\ln 3 + \frac{x}{2} \ln x \right]$$
$$= \lim_{x \to 0^+} \left[\ln 3 + \frac{\ln x}{2/x} \right]$$
$$= \lim_{x \to 0^+} \ln 3 + \lim_{x \to 0^+} \frac{1/x}{-2/x^2}$$
$$= \lim_{x \to 0^+} \ln 3 - \lim_{x \to 0^+} \frac{x}{2}$$
$$= \ln 3$$

So, $\lim_{x \to 0^+} 3(x)^{x/2} = 3$.

(c)



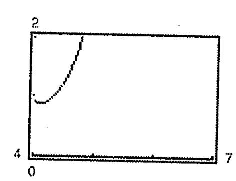
54. (a) $\lim_{x \to 4^+} \left[3(x - 4) \right]^{x-4} = 0^0$

(b) Let $y = \lim_{x \to 4^+} \left[3(x - 4) \right]^{x-4}$.

$$\ln y = \lim_{x \to 4^+} (x - 4) \ln\left[3(x - 4) \right]$$
$$= \lim_{x \to 4^+} \frac{\ln\left[3(x - 4) \right]}{1/(x - 4)}$$
$$= \lim_{x \to 4^+} \frac{1/(x - 4)}{-1/(x - 4)^2}$$
$$= \lim_{x \to 4^+} \left[-(x - 4) \right] = 0$$

So, $\lim_{x \to 4^+} \left[3(x - 4) \right]^{x-4} = 1$.

(c)



55. (a) $\lim\limits_{x\to1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = (\ln x)^{x-1}$.

$$\ln y = \ln\left[(\ln x)^{x-1}\right] = (x-1)\ln(\ln x)$$

$$= \frac{\ln(\ln x)}{(x-1)^{-1}}$$

$$\lim_{x\to1^+} \ln y = \lim_{x\to1^+} \frac{\ln(\ln x)}{(x-1)^{-1}}$$

$$= \lim_{x\to1^+} \frac{1/(x \ln x)}{-(x-1)^{-2}}$$

$$= \lim_{x\to1^+} \frac{-(x-1)^2}{x \ln x}$$

$$= \lim_{x\to1^+} \frac{-2(x-1)}{1 + \ln x} = 0$$

Because $\lim\limits_{x\to1^+} \ln y = 0$, $\lim\limits_{x\to1^+} y = 1$.

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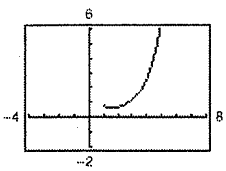


56. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

(a) $\lim\limits_{x\to0^+} \left[\cos\left(\dfrac{\pi}{2} - x\right)\right]^x = \lim\limits_{x\to0^+} [\sin x]^x = 0^0$

(b) Let $y = (\sin x)^x$

$$\ln y = x \ln(\sin x) = \frac{\ln(\sin x)}{1/x}$$

$$\lim_{x\to0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x\to0^+} \frac{\cos x/\sin x}{-1/x^2}$$

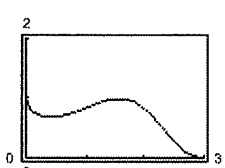
$$= \lim_{x\to0^+} \frac{-x^2 \cos x}{\sin x}$$

$$= \lim_{x\to0^+} \frac{x}{\sin x}\left(\frac{-x \cos x}{1}\right)$$

$$= 0$$

So, $\lim\limits_{x\to0^+} \left[\cos\left(\dfrac{\pi}{2} - x\right)\right]^x = 1$.

(c)



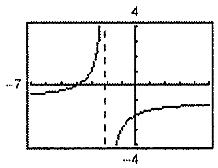
57. (a) $\lim\limits_{x\to2^+} \left(\dfrac{8}{x^2 - 4} - \dfrac{x}{x-2}\right) = \infty - \infty$

(b) $$\lim_{x\to2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x-2}\right) = \lim_{x\to2^+} \frac{8 - x(x+2)}{x^2 - 4}$$

$$= \lim_{x\to2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)}$$

$$= \lim_{x\to2^+} \frac{-(x+4)}{x+2} = \frac{-3}{2}$$

(c)



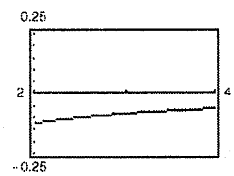
58. (a) $\lim\limits_{x\to2^+} \left(\dfrac{1}{x^2 - 4} - \dfrac{\sqrt{x-1}}{x^2 - 4}\right) = \infty - \infty$

(b) $$\lim_{x\to2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4}\right) = \lim_{x\to2^+} \frac{1 - \sqrt{x-1}}{x^2 - 4}$$

$$= \lim_{x\to2^+} \frac{-1/\left(2\sqrt{x-1}\right)}{2x}$$

$$= \lim_{x\to2^+} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{8}$$

(c)

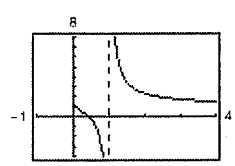


59. (a) $\lim\limits_{x\to1^+} \left(\dfrac{3}{\ln x} - \dfrac{2}{x-1}\right) = \infty - \infty$

(b) $$\lim_{x\to1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to1^+} \frac{3x - 3 - 2\ln x}{(x-1)\ln x}$$

$$= \lim_{x\to1^+} \frac{3 - (2/x)}{\left[(x-1)/x\right] + \ln x} = \infty$$

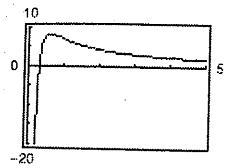
(c)



60. (a) $\lim_{x\to0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right)=\infty-\infty$

(b) $\lim_{x\to0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right)=\lim_{x\to0^+}\left(\frac{10x-3}{x^2}\right)=-\infty$

(c)



61. $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 1^\infty, 0^0, \infty-\infty, \infty^0$

62. See Theorem 8.4.

63. (a) Let $f(x)=x^2-25$ and $g(x)=x-5$.

(b) Let $f(x)=(x-5)^2$ and $g(x)=x^2-25$.

(c) Let $f(x)=x^2-25$ and $g(x)=(x-5)^3$.

(Answers will vary.)

64. Let $f(x)=x+25$ and $g(x)=x$.

(Answers will vary.)

65. (a) Yes: $\frac{0}{0}$

(b) No: $\frac{0}{-1}$

(c) Yes: $\frac{\infty}{\infty}$

(d) Yes: $\frac{0}{0}$

(e) No: $\frac{-1}{0}$

(f) Yes: $\frac{0}{0}$

66. (a) From the graph, $\lim_{x\to1^-}f(x)=\infty$.

(b) From the graph, $\lim_{x\to1^+}f(x)=-\infty$.

(c) From the graph, $\lim_{x\to1}f(x)$ does not exist.

67.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

68.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

69. $\lim_{x\to\infty}\frac{x^2}{e^{5x}}=\lim_{x\to\infty}\frac{2x}{5e^{5x}}=\lim_{x\to\infty}\frac{2}{25e^{5x}}=0$

70. $\lim_{x\to\infty}\frac{x^3}{e^{2x}}=\lim_{x\to\infty}\frac{3x^2}{2e^{2x}}=\lim_{x\to\infty}\frac{6x}{4e^{2x}}=\lim_{x\to\infty}\frac{6}{8e^{2x}}=0$

71. $$\lim_{x\to\infty}\frac{(\ln x)^3}{x}=\lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{1}$$
$$=\lim_{x\to\infty}\frac{3(\ln x)^2}{x}$$
$$=\lim_{x\to\infty}\frac{6(\ln x)(1/x)}{1}$$
$$=\lim_{x\to\infty}\frac{6(\ln x)}{x}=\lim_{x\to\infty}\frac{6}{x}=0$$

72. $$\lim_{x\to\infty}\frac{(\ln x)^2}{x^3}=\lim_{x\to\infty}\frac{(2\ln x)/x}{3x^2}$$
$$=\lim_{x\to\infty}\frac{2\ln x}{3x^3}$$
$$=\lim_{x\to\infty}\frac{2/x}{9x^2}=\lim_{x\to\infty}\frac{2}{9x^3}=0$$

73. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$

$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m}$

$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m}$

$= \cdots = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$

74. $\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$

$= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$

$= \cdots = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$

75. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 49.)

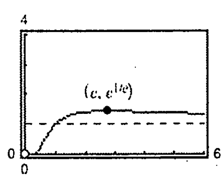
$\ln y = \frac{1}{x} \ln x$

$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$

$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$

Critical number:	$x = e$	
Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	+	–
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $(e, e^{1/e})$



76. $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

No horizontal asymptotes

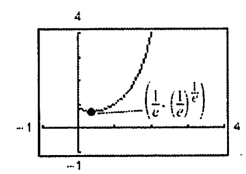
$\ln y = x \ln x$

$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$

$\frac{dy}{dx} = x^x(1 + \ln x) = 0$

Critical number:	$x = e^{-1}$	
Intervals:	$(0, e^{-1})$	$(e^{-1}, 0)$
Sign of dy/dx:	–	+
$y = f(x)$:	Decreasing	Increasing

Relative maximum: $\left(e^{-1}, (e^{-1})^{e^{-1}}\right) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$



77. $y = 2xe^{-x}$

$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

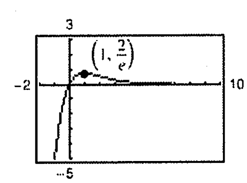
Horizontal asymptote: $y = 0$

$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$

$= 2e^{-x}(1 - x) = 0$

Critical number:	$x = 1$	
Intervals:	$(-\infty, 1)$	$(1, \infty)$
Sign of dy/dx:	+	–
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$



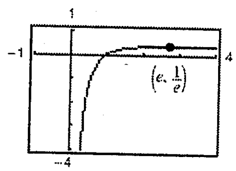
78. $y = \dfrac{\ln x}{x}$

Horizontal asymptote: $y = 0$ (See Example 2.)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number:	$x = e$	
Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(e, \dfrac{1}{e}\right)$



79. $\displaystyle\lim_{x\to 2} \frac{3x^2 + 4x + 1}{x^2 - x - 2} = \frac{21}{0}$

Limit is not of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

L'Hôpital's Rule does not apply.

80. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞.

L'Hôpital's Rule does not apply.

81. $\displaystyle\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or ∞/∞.

L'Hôpital's Rule does not apply.

82. $\displaystyle\lim_{x\to\infty} x \cos\frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞.

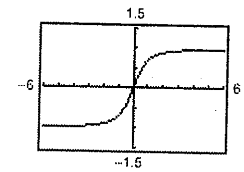
L'Hôpital's Rule does not apply.

83. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2+1}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$
$$= \lim_{x\to\infty} \frac{x/\sqrt{x^2+1}}{1}$$
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

(b) $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + 1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

(c)



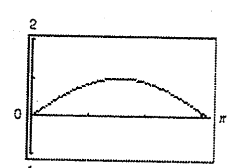
84. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$\displaystyle\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$ is indeterminant: $\dfrac{\infty}{\infty}$

$$\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} = \lim_{x\to\pi/2^-} \frac{\sec^2 x}{\sec x \tan x}$$
$$= \lim_{x\to\pi/2^-} \frac{\sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x\to\pi/2^-} \frac{\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$$

(b) $$\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x} = \lim_{x\to\pi/2^-} \frac{\sin x}{\cos x}(\cos x)$$
$$= \lim_{x\to\pi/2^-} \sin x = 1$$

(c)



85. $f(x) = \sin(3x),\ g(x) = \sin(4x)$

$f'(x) = 3\cos(3x),\ g'(x) = 4\cos(4x)$

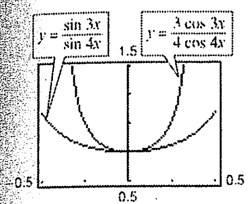
$$y_1 = \frac{f(x)}{g(x)} = \frac{\sin 3x}{\sin 4x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = \frac{3\cos 3x}{4\cos 4x}$$

As $x \to 0$, $y_1 \to 0.75$ and $y_2 \to 0.75$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$$



86. $f(x) = e^{3x} - 1,\ g(x) = x$

$f'(x) = 3e^{3x},\ g'(x) = 1$

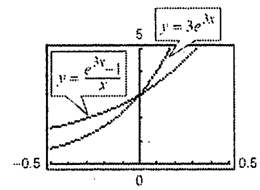
$$y_1 = \frac{f(x)}{g(x)} = \frac{e^{3x} - 1}{x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = 3e^{3x}$$

As $x \to 0$, $y_1 \to 3$ and $y_2 \to 3$

By L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{e^{3x} - 1}{x} = \lim_{x\to 0} \frac{3e^{3x}}{1} = 3$$



87. $$\lim_{k\to 0} \frac{32\left(1 - e^{-kt} + \dfrac{v_0 k e^{-kt}}{32}\right)}{k} = \lim_{k\to 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k\to 0} \left(v_0 e^{-kt}\right) = \lim_{k\to 0} \frac{32(0 + te^{-kt})}{1} + \lim_{k\to 0} \left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

88. $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\frac{r}{n^2}\left(\frac{1}{1 + (r/n)}\right)}{-\left(\frac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \frac{r}{n}}\right)\right] = rt$$

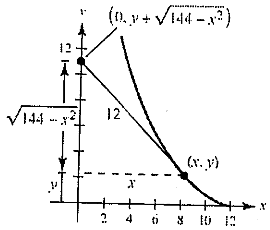
Because $\lim_{n\to\infty} \ln A = \ln P + rt$, you have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P} e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

89. Let N be a fixed value for n. Then

$$\lim_{x\to\infty} \frac{x^{N-1}}{e^x} = \lim_{x\to\infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \cdots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0.$$ (See Exercise 74.)

90. (a) $m = \dfrac{dy}{dx} = \dfrac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0} = -\dfrac{\sqrt{144 - x^2}}{x}$



(b) $y = -\displaystyle\int \frac{\sqrt{144 - x^2}}{x}\,dx$

Let $x = 12 \sin\theta$, $dx = 12\cos\theta\,d\theta$, $\sqrt{144 - x^2} = 12\cos\theta$.

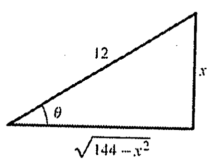
$$y = -\int \frac{12\cos\theta}{12\sin\theta}\,12\cos\theta\,d\theta = -12\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta$$

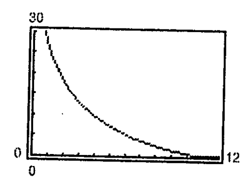
$$= -12\int(\csc\theta - \sin\theta)\,d\theta = -12\ln\left|\csc\theta - \cot\theta\right| - 12\cos\theta + C$$

$$= -12\ln\left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C = -12\ln\left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. So, $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

Note: $\dfrac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \le 12$





(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$

So,
$$\begin{aligned}
12 - \sqrt{144 - x^2} &= -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2} \\
-1 &= \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) \\
xe^{-1} &= 12 - \sqrt{144 - x^2} \\
\left(xe^{-1} - 12\right)^2 &= \left(-\sqrt{144 - x^2}\right)^2 \\
x^2e^{-2} - 24xe^{-1} + 144 &= 144 - x^2 \\
x^2\left(e^{-2} + 1\right) - 24xe^{-1} &= 0 \\
x\left[x\left(e^{-2} + 1\right) - 24e^{-1}\right] &= 0 \\
x &= 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.
\end{aligned}$$

Therefore, $s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\, dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + \left(144 - x^2\right)}{x^2}}\, dx$

$= \int_{7.77665}^{12} \frac{12}{x}\, dx = \Big[12 \ln|x|\Big]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2$ meters.

91. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\begin{aligned}
\frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\
\frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\
\frac{1}{1} &= \frac{3c}{2} \\
c &= \frac{2}{3}
\end{aligned}$$

92. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\begin{aligned}
\frac{f(2) - f(1)}{g(2) - g(1)} &= \frac{f'(c)}{g'(c)} \\
\frac{-1/2}{3} &= \frac{-1/c^2}{2c} \\
-\frac{1}{6} &= -\frac{1}{2c^3} \\
2c^3 &= 6 \\
c &= \sqrt[3]{3}
\end{aligned}$$

93. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

$$\begin{aligned}
\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\
\frac{1}{-1} &= \frac{\cos c}{-\sin c} \\
-1 &= -\cot c \\
c &= \frac{\pi}{4}
\end{aligned}$$

94. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\begin{aligned}
\frac{f(4) - f(1)}{g(4) - g(1)} &= \frac{f'(c)}{g'(c)} \\
\frac{\ln 4}{63} &= \frac{1/c}{3c^2} = \frac{1}{3c^3} \\
3c^3 \ln 4 &= 63 \\
c^3 &= \frac{21}{\ln 4} \\
c &= \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474
\end{aligned}$$

95. False. L'Hôpital's Rule does not apply because

$\lim_{x \to 0} \left(x^2 + x + 1\right) \neq 0.$

$$\lim_{x \to 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \to 0^+} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

114. (a) $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{x + x\sin x}{x^2 - 4} = \lim_{x\to\infty} \dfrac{1 + \sin x}{x - 4/x} = 0$

$\left(\text{Because } |1 + \sin x| \le 1 \text{ and } x \to \infty.\right)$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(1 + \sin x) = \infty$

$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(x^2 - 4\right) = \infty$

(c) $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = \lim_{x\to\infty} \dfrac{1 + \sin x + x\cos x}{2x}$ undefined

(d) No. If $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ does not exist, then you cannot assume anything about $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.

115. Let $f(x) = \left[\dfrac{1}{x} \cdot \dfrac{a^x - 1}{a - 1}\right]^{1/x}$.

For $a > 1$ and $x > 0$,

$$\ln f(x) = \frac{1}{x}\left[\ln\frac{1}{x} + \ln\left(a^x - 1\right) - \ln(a - 1)\right] = -\frac{\ln x}{x} + \frac{\ln\left(a^x - 1\right)}{x} - \frac{\ln(a - 1)}{x}.$$

As $x \to \infty$, $\dfrac{\ln x}{x} \to 0$, $\dfrac{\ln(a - 1)}{x} \to 0$, and $\dfrac{\ln\left(a^x - 1\right)}{x} = \dfrac{\ln\left[\left(1 - a^{-x}\right)a^x\right]}{x} = \dfrac{\ln\left(1 - a^{-x}\right)}{x} + \ln a \to \ln a$.

So, $\ln f(x) \to \ln a$.

For $0 < a < 1$ and $x > 0$,

$$\ln f(x) = \frac{-\ln x}{x} + \frac{\ln\left(1 - a^x\right)}{x} - \frac{\ln(1 - a)}{x} \to 0 \text{ as } x \to \infty.$$

Combining these results, $\lim_{x\to\infty} f(x) = \begin{cases} a & \text{if } a > 1 \\ 1 & \text{if } 0 < a < 1 \end{cases}$.

Section 8.8 Improper Integrals

1. $\int_0^1 \dfrac{dx}{5x - 3}$ is improper because $5x - 3 = 0$ when $x = \dfrac{3}{5}$, and $0 \le \dfrac{3}{5} \le 1$.

2. $\int_1^2 \dfrac{dx}{x^3}$ is not improper because $f(x) = \dfrac{1}{x^3}$ is continuous on $[1, 2]$.

3. $\int_0^1 \dfrac{2x - 5}{x^2 - 5x + 6}\,dx = \int_0^1 \dfrac{2x - 5}{(x - 2)(x - 3)}\,dx$ is not improper because $\dfrac{2x - 5}{(x - 2)(x - 3)}$ is continuous on $[0, 1]$.

4. $\int_1^\infty \ln\left(x^2\right) dx$ is improper because the upper limit of integration is ∞.

5. $\int_0^2 e^{-x}\,dx$ is not improper because $f(x) = e^{-x}$ is continuous on $[0, 2]$.

6. $\int_0^\infty \cos x\,dx$ is improper because the upper limit of integration is ∞.

7. $\int_{-\infty}^\infty \dfrac{\sin x}{4 + x^2}\,dx$ is improper because the limits of integration are $-\infty$ and ∞.

8. $\int_0^{\pi/4} \csc x\,dx$ is improper because $f(x) = \csc x$ is undefined at $x = 0$.

9. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+} \int_b^4 \frac{1}{\sqrt{x}}\,dx$$
$$= \lim_{b\to 0^+}\Big[2\sqrt{x}\Big]_b^4$$
$$= \lim_{b\to 0^+}\left(4 - 2\sqrt{b}\right) = 4$$

Converges

10. Infinite discontinuity at $x = 3$.

$$\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx = \lim_{b\to 3^+}\int_b^4 (x-3)^{-3/2}\,dx$$
$$= \lim_{b\to 3^+}\Big[-2(x-3)^{-1/2}\Big]_b^4$$
$$= \lim_{b\to 3^+}\left[-2 + \frac{2}{\sqrt{b-3}}\right] = \infty$$

Diverges

11. Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+}\left[-\frac{1}{x-1}\right]_c^2$$
$$= (\infty - 1) + (-1 + \infty)$$

Diverges

12. Infinite limit of integration.

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{b\to -\infty}\int_b^0 e^{3x}\,dx$$
$$= \lim_{b\to -\infty}\Big[\tfrac{1}{3}e^{3x}\Big]_b^0$$
$$= \lim_{b\to -\infty}\Big[\tfrac{1}{3} - \tfrac{1}{3}e^{3b}\Big] = \tfrac{1}{3}$$

Converges

13. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx \neq -2$

because the integrand is not defined at $x = 0$. The integral diverges.

14. $\displaystyle\int_{-2}^2 \frac{-2}{(x-1)^3}\,dx \neq \frac{8}{9}$

because the integral is not defined at $x = 1$. The integral diverges.

15. $\displaystyle\int_0^\infty e^{-x}\,dx \neq 0$. You need to evaluate the limit.

$$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b$$
$$= \lim_{b\to\infty}\Big[-e^{-b} + 1\Big] = 1$$

16. $\displaystyle\int_0^\pi \sec x\,dx \neq 0$ because $\sec x$ is not defined at $x = \pi/2$.

The integral diverges.

17.
$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b x^{-3}\,dx$$
$$= \lim_{b\to\infty}\left[\frac{x^{-2}}{-2}\right]_1^b$$
$$= \lim_{b\to\infty}\left[\frac{-1}{2b^2} + \frac{1}{2}\right] = \frac{1}{2}$$

18.
$$\int_1^\infty \frac{6}{x^4}\,dx = \lim_{b\to\infty} 6\int_1^b x^{-4}\,dx$$
$$= \lim_{b\to\infty} 6\left[\frac{x^{-3}}{-3}\right]_1^b$$
$$= \lim_{b\to\infty}\left[\frac{-2}{b^3} + 2\right] = 2$$

19.
$$\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-1/3}\,dx$$
$$= \lim_{b\to\infty}\left[\frac{9}{2}x^{2/3}\right]_1^b = \infty$$

Diverges

20.
$$\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b 4x^{-1/4}\,dx$$
$$= \lim_{b\to\infty}\left[\frac{16}{3}x^{3/4}\right]_1^b = \infty \quad \text{Diverges}$$

21.
$$\int_{-\infty}^0 xe^{-4x}\,dx = \lim_{b\to -\infty}\int_b^0 xe^{-4x}\,dx$$
$$= \lim_{b\to -\infty}\left[\left(\frac{-x}{4} - \frac{1}{16}\right)e^{-4x}\right]_b^0 \quad \text{(Integration by parts)}$$
$$= \lim_{b\to -\infty}\left[-\frac{1}{16} + \frac{b}{4} + \frac{1}{16}e^{-4b}\right] = -\infty$$

Diverges

22. $\int_0^\infty xe^{-x/3}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x/3}\,dx$

$= \lim_{b\to\infty}\left[(-3x-9)e^{-x/3}\right]_0^b$

$= \lim_{b\to\infty}\left[(-3b-9)e^{-b/3}+9\right] = 9$

23. $\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx$

$= \lim_{b\to\infty}\left[-e^{-x}\left(x^2+2x+2\right)\right]_0^b$

$= \lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b}+2\right) = 2$

Because $\lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

24. $\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\frac{1}{2}\left[e^{-x}(-\cos x+\sin x)\right]_0^b$

$= \frac{1}{2}\left[0-(-1)\right] = \frac{1}{2}$

25. $\int_4^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_4^b (\ln x)^{-3}\frac{1}{x}\,dx$

$= \lim_{b\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]_4^b$

$= \lim_{b\to\infty}\left[-\frac{1}{2}(\ln b)^{-2}+\frac{1}{2}(\ln 4)^{-2}\right]$

$= \frac{1}{2}\frac{1}{(2\ln 2)^2} = \frac{1}{2(\ln 4)^2}$

26. $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx$

$= \lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b = \infty$

Diverges

27. $\int_{-\infty}^\infty \frac{4}{16+x^2}\,dx = \int_{-\infty}^0 \frac{4}{16+x^2}\,dx + \int_0^\infty \frac{4}{16+x^2}\,dx$

$= \lim_{b\to-\infty}\int_b^0 \frac{4}{16+x^2}\,dx + \lim_{c\to\infty}\int_0^c \frac{4}{16+x^2}\,dx$

$= \lim_{b\to-\infty}\left[\arctan\left(\frac{x}{4}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\left(\frac{x}{4}\right)\right]_0^c$

$= \lim_{b\to-\infty}\left[0-\arctan\left(\frac{b}{4}\right)\right] + \lim_{c\to\infty}\left[\arctan\left(\frac{c}{4}\right)-0\right]$

$= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi$

28. $\int_0^\infty \frac{x^3}{\left(x^2+1\right)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\,dx - \lim_{b\to\infty}\int_0^b \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{2\left(x^2+1\right)}\right]_0^b = \infty - \frac{1}{2}$

Diverges

29. $\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}\,dx$

$= \lim_{b\to\infty}\left[\arctan\left(e^x\right)\right]_0^b$

$= \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$

30. $\int_0^\infty \frac{e^x}{1+e^x}\,dx = \lim_{b\to\infty}\left[\ln\left(1+e^x\right)\right]_0^b = \infty - \ln 2$

Diverges

31. $\int_0^\infty \cos \pi x\,dx = \lim_{b\to\infty}\left[\frac{1}{\pi}\sin \pi x\right]_0^b$

Diverges because $\sin \pi b$ does not approach a limit as $b \to \infty$.

32. $\int_0^\infty \sin\frac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\frac{x}{2}\right]_0^b$

Diverges because $\cos\frac{x}{2}$ does not approach a limit as $x \to \infty$.

33. $\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to0^+}\left[\frac{-1}{x}\right]_b^1 = \lim_{b\to0^+}\left(-1+\frac{1}{b}\right) = -1+\infty$

Diverges

34. $\int_0^5 \frac{10}{x}\,dx = \lim_{b\to 0^+} \int_b^5 \frac{10}{x}\,dx$

$= \lim_{b\to 0^+} \left[10 \ln x\right]_b^5$

$= \lim_{b\to 0^+} \left(10 \ln 5 - 10 \ln b\right) = \infty$

Diverges

35. $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{b\to 1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^b + \lim_{c\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$

36. $\int_0^8 \frac{3}{\sqrt{8-x}}\,dx = \lim_{b\to 8^-} 3\int_0^b (8-x)^{-1/2}\,dx$

$= \lim_{b\to 8^-}\left[-6\sqrt{8-x}\right]_0^b$

$= \lim_{b\to 8^-}\left(-6\sqrt{8-b} + 6\sqrt{8}\right)$

$= 12\sqrt{2}$

37. $\int_0^1 x \ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1$

$= \lim_{b\to 0^+}\left(\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4}\right) = \frac{-1}{4}$

because $\lim_{b\to 0^+}\left(b^2 \ln b\right) = 0$ by L'Hôpital's Rule.

38. $\int_0^e \ln x^2\,dx = \lim_{b\to 0^+}\int_0^e 2\ln x\,dx$

$= \lim_{b\to 0^+}\left[2x\ln x - 2x\right]_b^e$

$= \lim_{b\to 0^+}\left[(2e - 2e) - (2b\ln b - 2b)\right]$

$= 0$

39. $\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to(\pi/2)^-}\left[\ln|\sec\theta|\right]_0^b = \infty$

Diverges

40. $\int_0^{\pi/2} \sec\theta\,d\theta = \lim_{b\to(\pi/2)}\left[\ln|\sec\theta + \tan\theta|\right]_0^b = \infty$

Diverges

41. $\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx = \lim_{b\to 2^+}\int_b^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

$= \lim_{b\to 2^+}\left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$

$= \lim_{b\to 2^+}\left(\operatorname{arcsec} 2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$

$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$

42. $\int_3^6 \frac{1}{\sqrt{36-x^2}}\,dx = \lim_{b\to 6^-}\int_3^b \frac{1}{\sqrt{36-x^2}}\,dx$

$= \lim_{b\to 6^-}\left[\arcsin\frac{x}{6}\right]_3^b$

$= \lim_{b\to 6^-}\left[\arcsin\frac{b}{6} - \arcsin\frac{1}{2}\right]$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

43. $\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{b\to 3^+}\left[\ln\left|x + \sqrt{x^2-9}\right|\right]_b^5$

$= \lim_{b\to 3^+}\left[\ln 9 - \ln\left(b + \sqrt{b^2-9}\right)\right]$

$= \ln 9 - \ln 3$

$= \ln\frac{9}{3} = \ln 3$

44. $\int_0^5 \frac{1}{25-x^2}\,dx = \lim_{b\to 5^-}\int_0^b \frac{1}{25-x^2}\,dx$

$= \lim_{b\to 5^-}\int_0^b \frac{1}{10}\left(\frac{1}{x+5} - \frac{1}{x-5}\right)dx$ (partial fractions)

$= \lim_{b\to 5^-}\left[\frac{1}{10}\ln\left|\frac{x+5}{x-5}\right|\right]_0^b$

$= \infty - 0$ Diverges

45. $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx = \lim_{b\to3^+}\int_b^5 \frac{1}{x\sqrt{x^2-9}}\,dx + \lim_{c\to\infty}\int_5^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$

$$= \lim_{b\to3^+}\left[\frac{1}{3}\operatorname{arcsec}\frac{x}{3}\right]_b^5 + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{x}{3}\right)\right]_5^\infty$$

$$= \lim_{b\to3^+}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{b}{3}\right)\right] + \lim_{c\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\left(\frac{c}{3}\right) - \frac{1}{3}\operatorname{arcsec}\left(\frac{5}{3}\right)\right] = -0 + \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

46. $\displaystyle\int_4^\infty \frac{\sqrt{x^2-16}}{x^2}\,dx = \lim_{b\to\infty}\int_4^b \frac{\sqrt{x^2-16}}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{-\sqrt{x^2-16}}{x} + \ln\left|x + \sqrt{x^2-16}\right|\right]_4^b \quad \text{(Formula 30)}$$

$$= \lim_{b\to\infty}\left[-\frac{\sqrt{b^2-16}}{b} + \ln\left|b + \sqrt{b^2-16}\right| - \ln 4\right] = \infty$$

Diverges

47. $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)}\,dx + \int_1^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)}\,dx = \int \frac{4(2u\,du)}{u(u^2+6)} = 8\int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

So, $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \lim_{b\to0^+}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$

$$= \left[\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}(0)\right] + \left[\frac{8}{\sqrt{6}}\left(\frac{\pi}{2}\right) - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right] = \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

48. $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln\left|\ln|x|\right| + C$

So,

$$\int_1^\infty \frac{1}{x\ln x}\,dx = \int_1^e \frac{1}{x\ln x}\,dx + \int_e^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to1^+}\left[\ln(\ln x)\right]_1^e + \lim_{c\to\infty}\left[\ln(\ln x)\right]_e^\infty.$$

Diverges

49. If $p = 1$, $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_1^b$

$$= \lim_{b\to\infty}(\ln b) = \infty.$$

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right).$$

This converges to $\dfrac{1}{p-1}$ if $1 - p < 0$ or $p > 1$.

50. If $p = 1$, $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a\to0^+}\left[\ln x\right]_a^1 = \lim_{a\to0^+} -\ln a = \infty.$

Diverges. For $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right).$$

This converges to $\dfrac{1}{1-p}$ if $1 - p > 0$ or $p < 1$.

51. For $n = 1$:

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-e^{-x}x - e^{-x}\right]_0^b \qquad \left(\text{Parts: } u = x,\, dv = e^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-e^{-b}b - e^{-b} + 1\right)$$
$$= \lim_{b\to\infty}\left(\frac{-b}{e^b} - \frac{1}{e^b} + 1\right) = 1 \quad \text{(L'Hôpital's Rule)}$$

Assume that $\int_0^\infty x^n e^{-x}\,dx$ converges. Then for $n + 1$ you have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts $\left(u = x^{n+1},\, du = (n+1)x^n\,dx,\, dv = e^{-x}\,dx,\, v = -e^{-x}\right)$.

So,

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\left[-x^{n+1}e^{-x}\right]_0^b + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)\int_0^\infty x^n e^{-x}\,dx, \text{ which converges.}$$

52. (a) $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-b}\right]_1^b = 1$

Because $e^{-x^2} \le e^{-x}$ on $[1, \infty)$

and

$$\int_1^\infty e^{-x}\,dx$$

converges, then so does

$$\int_1^\infty e^{-x^2}\,dx.$$

(b) $\int_1^\infty \frac{1}{x^5}\,dx$ converges (see Exercise 49).

Because $\frac{1}{x^5+1} < \frac{1}{x^5}$ on $[1, \infty)$, then $\int_1^\infty \frac{1}{x^5+1}\,dx$ also converges.

53. $\int_0^1 \frac{1}{x^5}\,dx$ diverges by Exercise 50. $(p = 5)$

54. $\int_0^1 \frac{1}{x^{1/5}}\,dx$ converges by Exercise 50. $\left(p = \frac{1}{5}\right)$

55. $\int_1^\infty \frac{1}{x^5}\,dx$ converges by Exercise 49. $(p = 5)$

56. $\int_0^\infty x^4 e^{-x}\,dx$ converges by Exercise 51. $(n = 4)$

57. Because $\frac{1}{x^2+5} \le \frac{1}{x^2}$ on $[1, \infty)$ and

$\int_1^\infty \frac{1}{x^2}\,dx$ converges by Exercise 49,

$\int_1^\infty \frac{1}{x^2+5}\,dx$ converges.

58. Because $\frac{1}{\sqrt{x-1}} \ge \frac{1}{x}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 55, $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ diverges.

59. Because $\frac{1}{\sqrt[3]{x(x-1)}} \ge \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and

$\int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$ diverges by Exercise 49,

$\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

60. Because $\frac{1}{\sqrt{x}(1+x)} \le \frac{1}{x^{3/2}}$ on $[1, \infty)$ and

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by Exercise 49,

$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

61. $\int_1^\infty \frac{2}{x^2}\,dx$ converges, and $\frac{1-\sin x}{x^2} \le \frac{2}{x^2}$ on $[1, \infty)$, so $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges.

62. $\int_0^\infty \frac{1}{e^x}\,dx = \int_0^\infty e^{-x}\,dx$ converges, and $\frac{1}{e^x} \ge \frac{1}{e^x + x}$ on $[0, \infty)$, so $\int_0^\infty \frac{1}{e^x + x}\,dx$ converges.

63. Answers will vary. *Sample answer:*

An integral with infinite integration limits or an integral with an infinite discontinuity at or between the integration limits

64. When the limit of the integral exists, the improper integral converges. When the limit does not exist, the improper integral diverges.

65. $\int_{-1}^1 \frac{1}{x^3}\,dx = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 50.

66. $\frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

67.
$$\begin{aligned} A &= \int_{-\infty}^1 e^x\,dx \\ &= \lim_{b\to-\infty} \int_b^1 e^x\,dx \\ &= \lim_{b\to-\infty} \left[e^x\right]_b^1 \\ &= \lim_{b\to-\infty} \left(e - e^b\right) = e \end{aligned}$$

68.
$$\begin{aligned} A &= \int_0^1 -\ln x\,dx \\ &= -\lim_{b\to0^+} \int_b^1 \ln x\,dx \\ &= -\lim_{b\to0^+} \left[x\ln x - x\right]_b^1 \\ &= -\lim_{b\to0^+} \left[(0-1) - b\ln b + b\right] \\ &= 1 \end{aligned}$$

Note: $\lim_{b\to0^+} b\ln b = \lim_{b\to0^+} \frac{\ln b}{1/b} = \lim_{b\to0^+} \frac{1/b}{-1/b^2} = 0$

69.
$$\begin{aligned} A &= \int_{-\infty}^\infty \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to-\infty} \int_b^0 \frac{1}{x^2+1}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to-\infty} \left[\arctan(x)\right]_b^0 + \lim_{b\to\infty}\left[\arctan(x)\right]_0^b \\ &= \lim_{b\to-\infty} \left[0 - \arctan(b)\right] + \lim_{b\to\infty}\left[\arctan(b) - 0\right] \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \end{aligned}$$

70.
$$\begin{aligned} A &= \int_{-\infty}^\infty \frac{8}{x^2+4}\,dx \\ &= \lim_{b\to-\infty} \int_b^0 \frac{8}{x^2+4}\,dx + \lim_{b\to\infty} \int_0^b \frac{8}{x^2+4}\,dx \\ &= \lim_{b\to-\infty} \left[4\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{b\to\infty}\left[4\arctan\left(\frac{x}{2}\right)\right]_0^b \\ &= \lim_{b\to-\infty} \left[0 - 4\arctan\left(\frac{b}{2}\right)\right] + \lim_{b\to\infty}\left[4\arctan\left(\frac{b}{2}\right) - 0\right] \\ &= -4\left(\frac{-\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) = 4\pi \end{aligned}$$

71. (a)
$$\begin{aligned} A &= \int_0^\infty e^{-x}\,dx \\ &= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 0 - (-1) = 1 \end{aligned}$$

(b) **Disk:**
$$\begin{aligned} V &= \pi\int_0^\infty \left(e^{-x}\right)^2\,dx \\ &= \lim_{b\to\infty} \pi\left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{\pi}{2} \end{aligned}$$

(c) **Shell:**
$$\begin{aligned} V &= 2\pi\int_0^\infty xe^{-x}\,dx \\ &= \lim_{b\to\infty} 2\pi\left[-e^{-x}(x+1)\right]_0^b = 2\pi \end{aligned}$$

72. (a) $A = \int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

(b) **Disk:**
$$V = \pi\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**
$$V = 2\pi\int_1^\infty x\left(\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\left[2\pi(\ln x)\right]_1^b = \infty$$

Diverges

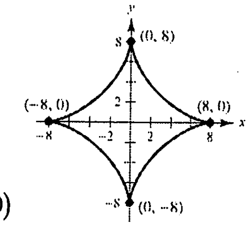
73. $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}, \quad (x > 0)$$

$$s = 4\int_0^8 \frac{2}{x^{1/3}}\,dx = \lim_{b \to 0^+}\left[8 \cdot \frac{3}{2}x^{2/3}\right]_b^8 = 48$$



74. $y = \sqrt{16 - x^2}, \quad 0 \le x \le 4$

$$y' = \frac{-x}{\sqrt{16 - x^2}}$$

$$s = \int_0^4 \sqrt{1 + \frac{x^2}{16 - x^2}}\,dx = \int_0^4 \frac{4}{\sqrt{16 - x^2}}\,dx = \lim_{t \to 4^-}\int_0^t \frac{4}{\sqrt{16 - x^2}}\,dx = \lim_{t \to 4^-}\left[4 \arcsin\left(\frac{x}{4}\right)\right]_0^t = \lim_{t \to 4^-} 4 \arcsin\left(\frac{t}{4}\right) = 2\pi$$

75. $(x - 2)^2 + y^2 = 1$

$$2(x - 2) + 2yy' = 0$$

$$y' = \frac{-(x - 2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left[(x - 2)^2/y^2\right]} = \frac{1}{y}\ (\text{Assume } y > 0.)$$

$$S = 4\pi\int_1^3 \frac{x}{y}\,dx = 4\pi\int_1^3 \frac{x}{\sqrt{1 - (x - 2)^2}}\,dx = 4\pi\int_1^3\left[\frac{x - 2}{\sqrt{1 - (x - 2)^2}} + \frac{2}{\sqrt{1 - (x - 2)^2}}\right]dx$$

$$= \lim_{\substack{a \to 1^+ \\ b \to 3^-}} 4\pi\left[-\sqrt{1 - (x - 2)^2} + 2\arcsin(x - 2)\right]_a^b = 4\pi\left[0 + 2\arcsin(1) - 2\arcsin(-1)\right] = 8\pi^2$$

76. $y = 2e^{-x}$

$$y' = -2e^{-x}$$

$$S = 2\pi\int_0^\infty \left(2e^{-x}\right)\sqrt{1 + 4e^{-2x}}\,dx$$

Let $u = e^{-x}$, $du = -e^{-x}\,dx$.

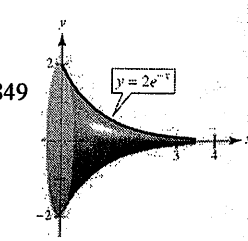
$$\int e^{-x}\sqrt{1 + 4e^{-2x}}\,dx = -\int\sqrt{1 + 4u^2}\,du$$

$$= -\tfrac{1}{4}\left[2u\sqrt{4u^2 + 1} + \ln\left|2u + \sqrt{4u^2 + 1}\right|\right] + C$$

$$= -\tfrac{1}{4}\left[2e^{-x}\sqrt{4e^{-2x} + 1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x} + 1}\right|\right] + C$$

$$S = 4\pi\lim_{b \to \infty}\int_0^b \left(e^{-x}\right)\sqrt{1 + 4e^{-2x}}\,dx$$

$$= -\pi\lim_{b \to \infty}\left[2e^{-x}\sqrt{4e^{-2x} + 1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x} + 1}\right|\right]_0^b = \pi\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 18.5849$$



77. (a) $F(x) = \frac{K}{x^2}, 5 = \frac{K}{(4000)^2}, K = 80{,}000{,}000$

$$W = \int_{4000}^{\infty} \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{b} = 20{,}000 \text{ mi-ton}$$

(b) $\frac{W}{2} = 10{,}000 = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{b} = \frac{-80{,}000{,}000}{b} + 20{,}000$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$
$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above the earth's surface.

78. (a) $F(x) = \frac{k}{x^2}, 10 = \frac{k}{4000^2}, k = 10(4000^2)$

$$W = \int_{4000}^{\infty} \frac{10(4000^2)}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-10(4000^2)}{x}\right]_{4000}^{b}$$
$$= \frac{10(4000^2)}{4000} = 40{,}000 \text{ mi-ton}$$

(b) $\frac{W}{2} = 20{,}000 = \left[\frac{-10(4000^2)}{x}\right]_{4000}^{b}$

$$= \frac{-10(4000^2)}{b} + 40{,}000$$
$$\frac{10(4000^2)}{b} = 20{,}000$$
$$b = 8000$$

Therefore, the rocket has traveled 4000 miles above the earth's surface.

79. (a) $\int_{-\infty}^{\infty} \frac{1}{7}e^{-t/7}\,dt = \int_{0}^{\infty} \frac{1}{7}e^{-t/7}\,dt = \lim_{b\to\infty}\left[-e^{-t/7}\right]_0^b = 1$

(b) $\int_0^4 \frac{1}{7}e^{-t/7}\,dt = \left[-e^{-t/7}\right]_0^4 = -e^{-4/7} + 1$

$\approx 0.4353 = 43.53\%$

(c) $\int_0^{\infty} t\left[\frac{1}{7}e^{-t/7}\right]dt = \lim_{b\to\infty}\left[-te^{-t/7} - 7e^{-t/7}\right]_0^b$

$= 0 + 7 = 7$

80. (a) $\int_{-\infty}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \int_{0}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \lim_{b\to\infty}\left[-e^{-2t/5}\right]_0^b = 1$

(b) $\int_0^4 \frac{2}{5}e^{-2t/5}\,dt = \left[-e^{-2t/5}\right]_0^4 = -e^{-8/5} + 1$

$\approx 0.7981 = 79.81\%$

(c) $\int_0^{\infty} t\left[\frac{2}{5}e^{-2t/5}\right]dt = \lim_{b\to\infty}\left[-te^{2t/5} - \frac{5}{2}e^{-2t/5}\right]_0^b = \frac{5}{2}$

81. (a) $C = 650{,}000 + \int_0^5 25{,}000\,e^{-0.06t}\,dt = 650{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^5 \approx \$757{,}992.41$

(b) $C = 650{,}000 + \int_0^{10} 25{,}000e^{-0.06t}\,dt \approx \$837{,}995.15$

(c) $C = 650{,}000 + \int_0^{\infty} 25{,}000e^{-0.06t}\,dt = 650{,}000 - \lim_{b\to\infty}\left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^b \approx \$1{,}066{,}666.67$

82. (a) $C = 650{,}000 + \int_0^5 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5 \approx \$778{,}512.58$$

(b) $C = 650{,}000 + \int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^{10} \approx \$905{,}718.14$$

(c) $C = 650{,}000 + \int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$$= 650{,}000 + 25{,}000\lim_{b\to\infty}\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b \approx \$1{,}622{,}222.22$$