8.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis **In Exercises 1–4, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to support your result.**

1. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 3x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

2. $\lim_{x\to 0} \dfrac{1-e^x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

3. $\lim_{x\to\infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

4. $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2-2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

Using Two Methods **In Exercises 5–10, evaluate the limit (a) using techniques from Chapters 1 and 3 and (b) using L'Hôpital's Rule.**

5. $\lim_{x\to 4} \dfrac{3(x-4)}{x^2-16}$

6. $\lim_{x\to -4} \dfrac{2x^2+13x+20}{x+4}$

7. $\lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6}$

8. $\lim_{x\to 0} \dfrac{\sin 6x}{4x}$

9. $\lim_{x\to\infty} \dfrac{5x^2-3x+1}{3x^2-5}$

10. $\lim_{x\to\infty} \dfrac{4x-3}{5x^2+1}$

Evaluating a Limit **In Exercises 11–42, evaluate the limit, using L'Hôpital's Rule if necessary.**

11. $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

12. $\lim_{x\to -2} \dfrac{x^2-3x-10}{x+2}$

13. $\lim_{x\to 0} \dfrac{\sqrt{25-x^2}-5}{x}$

14. $\lim_{x\to 5^-} \dfrac{\sqrt{25-x^2}}{x-5}$

15. $\lim_{x\to 0^+} \dfrac{e^x-(1+x)}{x^3}$

16. $\lim_{x\to 1} \dfrac{\ln x^3}{x^2-1}$

17. $\lim_{x\to 1} \dfrac{x^{11}-1}{x^4-1}$

18. $\lim_{x\to 1} \dfrac{x^a-1}{x^b-1}$, where $a, b \neq 0$

19. $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$

20. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$, where $a, b \neq 0$

21. $\lim_{x\to 0} \dfrac{\arcsin x}{x}$

22. $\lim_{x\to 1} \dfrac{\arctan x - (\pi/4)}{x-1}$

23. $\lim_{x\to\infty} \dfrac{5x^2+3x-1}{4x^2+5}$

24. $\lim_{x\to\infty} \dfrac{5x+3}{x^3-6x+2}$

25. $\lim_{x\to\infty} \dfrac{x^2+4x+7}{x-6}$

26. $\lim_{x\to\infty} \dfrac{x^3}{x+2}$

27. $\lim_{x\to\infty} \dfrac{x^3}{e^{x/2}}$

28. $\lim_{x\to\infty} \dfrac{x^3}{e^{x^2}}$

29. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$

30. $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x^2+1}}$

31. $\lim_{x\to\infty} \dfrac{\cos x}{x}$

32. $\lim_{x\to\infty} \dfrac{\sin x}{x-\pi}$

33. $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$

34. $\lim_{x\to\infty} \dfrac{\ln x^4}{x^3}$

35. $\lim_{x\to\infty} \dfrac{e^x}{x^4}$

36. $\lim_{x\to\infty} \dfrac{e^{x/2}}{x}$

37. $\lim_{x\to 0} \dfrac{\sin 5x}{\tan 9x}$

38. $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x}$

39. $\lim_{x\to 0} \dfrac{\arctan x}{\sin x}$

40. $\lim_{x\to 0} \dfrac{x}{\arctan 2x}$

41. $\lim_{x\to\infty} \dfrac{\int_1^x \ln(e^{4t-1})\,dt}{x}$

42. $\lim_{x\to 1^+} \dfrac{\int_1^x \cos\theta\,d\theta}{x-1}$

Evaluating a Limit **In Exercises 43–60, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).**

43. $\lim_{x\to\infty} x \ln x$

44. $\lim_{x\to 0^+} x^3 \cot x$

45. $\lim_{x\to\infty} \left(x \sin \dfrac{1}{x}\right)$

46. $\lim_{x\to\infty} x \tan \dfrac{1}{x}$

47. $\lim_{x\to 0^+} x^{1/x}$

48. $\lim_{x\to 0^+} (e^x + x)^{2/x}$

49. $\lim_{x\to\infty} x^{1/x}$

50. $\lim_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$

51. $\lim_{x\to 0^+} (1+x)^{1/x}$

52. $\lim_{x\to\infty} (1+x)^{1/x}$

53. $\lim_{x\to 0^+} [3(x)^{x/2}]$

54. $\lim_{x\to 4^+} [3(x-4)]^{x-4}$

55. $\lim_{x\to 1^+} (\ln x)^{x-1}$

56. $\lim_{x\to 0^+} \left[\cos\left(\dfrac{\pi}{2}-x\right)\right]^x$

57. $\lim_{x\to 2^+} \left(\dfrac{8}{x^2-4} - \dfrac{x}{x-2}\right)$

58. $\lim_{x\to 2^+} \left(\dfrac{1}{x^2-4} - \dfrac{\sqrt{x-1}}{x^2-4}\right)$

59. $\lim_{x\to 1^+} \left(\dfrac{3}{\ln x} - \dfrac{2}{x-1}\right)$

60. $\lim_{x\to 0^+} \left(\dfrac{10}{x} - \dfrac{3}{x^2}\right)$

WRITING ABOUT CONCEPTS

61. Indeterminate Forms List six different indeterminate forms.

62. L'Hôpital's Rule State L'Hôpital's Rule.

63. Finding Functions Find differentiable functions f and g that satisfy the specified condition such that

$$\lim_{x\to 5} f(x) = 0 \quad \text{and} \quad \lim_{x\to 5} g(x) = 0.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

64. Finding Functions Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} [f(x) - g(x)] = 25.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

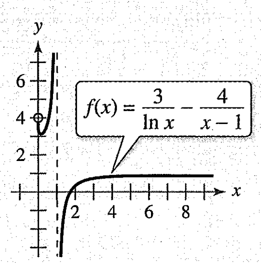
65. L'Hôpital's Rule Determine which of the following limits can be evaluated using L'Hôpital's Rule. Explain your reasoning. Do not evaluate the limit.

(a) $\lim_{x\to 2} \frac{x-2}{x^3 - x - 6}$ (b) $\lim_{x\to 0} \frac{x^2 - 4x}{2x - 1}$

(c) $\lim_{x\to\infty} \frac{x^3}{e^x}$ (d) $\lim_{x\to 3} \frac{e^{x^2} - e^9}{x - 3}$

(e) $\lim_{x\to 1} \frac{\cos \pi x}{\ln x}$ (f) $\lim_{x\to 1} \frac{1 + x(\ln x - 1)}{(x-1)\ln x}$

66. HOW DO YOU SEE IT? Use the graph of f to find the limit.



(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

67. Numerical Approach Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

68. Numerical Approach Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

Comparing Functions **In Exercises 69–74, use L'Hôpital's Rule to determine the comparative rates of increase of the functions $f(x) = x^m$, $g(x) = e^{nx}$, and $h(x) = (\ln x)^n$, where $n > 0$, $m > 0$, and $x \to \infty$.**

69. $\lim_{x\to\infty} \frac{x^2}{e^{5x}}$ **70.** $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

71. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$ **72.** $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$

73. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$ **74.** $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

Asymptotes and Relative Extrema **In Exercises 75–78, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function. (*Hint:* Some of the limits required in finding asymptotes have been found in previous exercises.)**

75. $y = x^{1/x}, \quad x > 0$ **76.** $y = x^x, \quad x > 0$

77. $y = 2xe^{-x}$ **78.** $y = \frac{\ln x}{x}$

Think About It **In Exercises 79–82, L'Hôpital's Rule is used incorrectly. Describe the error.**

79. $\lim_{x\to 2} \frac{3x^2 + 4x + 1}{x^2 - x - 2} = \lim_{x\to 2} \frac{6x + 4}{2x - 1} = \lim_{x\to 2} \frac{6}{2} = 3$

80. $\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2e^x = 2$

81. $\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \lim_{x\to\infty} \frac{-e^{-x}}{-e^{-x}} = \lim_{x\to\infty} 1 = 1$

82. $\lim_{x\to\infty} x \cos \frac{1}{x} = \lim_{x\to\infty} \frac{\cos(1/x)}{1/x} = \lim_{x\to\infty} \frac{[-\sin(1/x)](1/x^2)}{-1/x^2} = 0$

Analytical Approach **In Exercises 83 and 84, (a) explain why L'Hôpital's Rule cannot be used to find the limit, (b) find the limit analytically, and (c) use a graphing utility to graph the function and approximate the limit from the graph. Compare the result with that in part (b).**

83. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$ **84.** $\lim_{x\to\pi/2^-} \frac{\tan x}{\sec x}$

Graphical Analysis **In Exercises 85 and 86, graph $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$. What do you notice about these ratios as $x \to 0$? How does this illustrate L'Hôpital's Rule?**

85. $f(x) = \sin 3x, \quad g(x) = \sin 4x$

86. $f(x) = e^{3x} - 1, \quad g(x) = x$

87. Velocity in a Resisting Medium The velocity v of an object falling through a resisting medium such as air or water is given by

$$v = \frac{32}{k}\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)$$

where v_0 is the initial velocity, t is the time in seconds, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling body in a vacuum by fixing v_0 and t and letting k approach zero. (Assume that the downward direction is positive.)

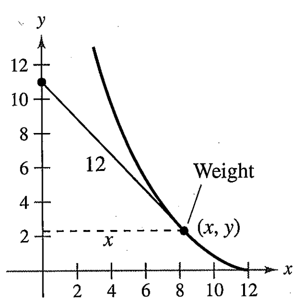
88. Compound Interest The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year approaches infinity is given by $A = Pe^{rt}$.

89. The Gamma Function The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function given by $f(x) = x^{n-1}e^{-x}, \quad n > 0$. Show that for any fixed value of n, the limit of $f(x)$ as x approaches infinity is zero.

90. Tractrix A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope (see figure). Initially, the weight is located at the point $(12, 0)$.



(a) Show that the slope of the tangent line of the path of the weight is

$$\frac{dy}{dx} = -\frac{\sqrt{144 - x^2}}{x}.$$

(b) Use the result of part (a) to find the equation of the path of the weight. Use a graphing utility to graph the path and compare it with the figure.

(c) Find any vertical asymptotes of the graph in part (b).

(d) When the person has reached the point $(0, 12)$, how far has the weight moved?

Extended Mean Value Theorem **In Exercises 91–94, apply the Extended Mean Value Theorem to the functions f and g on the given interval. Find all values c in the interval (a, b) such that**

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Functions	Interval
91. $f(x) = x^3, \quad g(x) = x^2 + 1$	$[0, 1]$
92. $f(x) = \frac{1}{x}, \quad g(x) = x^2 - 4$	$[1, 2]$
93. $f(x) = \sin x, \quad g(x) = \cos x$	$\left[0, \frac{\pi}{2}\right]$
94. $f(x) = \ln x, \quad g(x) = x^3$	$[1, 4]$

True or False? **In Exercises 95–98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

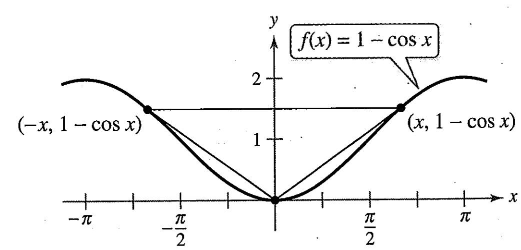
95. $\lim_{x\to 0}\left[\frac{x^2 + x + 1}{x}\right] = \lim_{x\to 0}\left[\frac{2x + 1}{1}\right] = 1$

96. If $y = \frac{e^x}{x^2}$, then $y' = \frac{e^x}{2x}$.

97. If $p(x)$ is a polynomial, then $\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$.

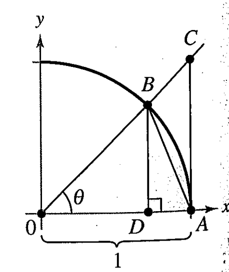
98. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

99. Area Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded area in the figure.



100. Finding a Limit In Section 1.3, a geometric argument (see figure) was used to prove that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$



(a) Write the area of $\triangle ABD$ in terms of θ.

(b) Write the area of the shaded region in terms of θ.

(c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.

Continuous Function **In Exercises 101 and 102, find the value of c that makes the function continuous at $x = 0$.**

101. $f(x) = \begin{cases} \dfrac{4x - 2\sin 2x}{2x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$

102. $f(x) = \begin{cases} (e^x + x)^{1/x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

103. Finding Values Find the values of a and b such that

$$\lim_{x \to 0} \frac{a - \cos bx}{x^2} = 2.$$

104. Evaluating a Limit Use a graphing utility to graph

$$f(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.1$, and 0.01. Then evaluate the limit

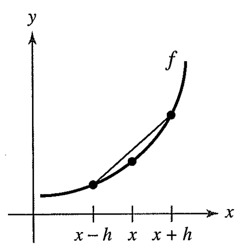
$$\lim_{k \to 0^+} \frac{x^k - 1}{k}.$$

105. Finding a Derivative

(a) Let $f'(x)$ be continuous. Show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).$$

(b) Explain the result of part (a) graphically.



106. Finding a Second Derivative Let $f''(x)$ be continuous. Show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).$$

107. Evaluating a Limit Consider the limit $\lim_{x \to 0^+} (-x \ln x)$.

(a) Describe the type of indeterminate form that is obtained by direct substitution.

(b) Evaluate the limit. Use a graphing utility to verify the result.

FOR FURTHER INFORMATION For a geometric approach to this exercise, see the article "A Geometric Proof of $\lim_{d \to 0^+} (-d \ln d) = 0$" by John H. Mathews in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

108. Proof Prove that if $f(x) \geq 0$, $\lim_{x \to a} f(x) = 0$, and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} f(x)^{g(x)} = 0$.

109. Proof Prove that if $f(x) \geq 0$, $\lim_{x \to a} f(x) = 0$, and $\lim_{x \to a} g(x) = -\infty$, then $\lim_{x \to a} f(x)^{g(x)} = \infty$.

110. Proof Prove the following generalization of the Mean Value Theorem. If f is twice differentiable on the closed interval $[a, b]$, then

$$f(b) - f(a) = f'(a)(b - a) - \int_a^b f''(t)(t - b)\, dt.$$

111. Indeterminate Forms Show that the indeterminate forms 0^0, ∞^0, and 1^∞ do not always have a value of 1 by evaluating each limit.

(a) $\lim_{x \to 0^+} x^{\ln 2/(1 + \ln x)}$

(b) $\lim_{x \to \infty} x^{\ln 2/(1 + \ln x)}$

(c) $\lim_{x \to 0} (x + 1)^{(\ln 2)/x}$

112. Calculus History In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

113. Finding a Limit Consider the function

$$h(x) = \frac{x + \sin x}{x}.$$

(a) Use a graphing utility to graph the function. Then use the *zoom* and *trace* features to investigate $\lim_{x \to \infty} h(x)$.

(b) Find $\lim_{x \to \infty} h(x)$ analytically by writing

$$h(x) = \frac{x}{x} + \frac{\sin x}{x}.$$

(c) Can you use L'Hôpital's Rule to find $\lim_{x \to \infty} h(x)$? Explain your reasoning.

114. Evaluating a Limit Let $f(x) = x + x \sin x$ and $g(x) = x^2 - 4$.

(a) Show that $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$.

(b) Show that $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = \infty$.

(c) Evaluate the limit

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

What do you notice?

(d) Do your answers to parts (a) through (c) contradict L'Hôpital's Rule? Explain your reasoning.

PUTNAM EXAM CHALLENGE

115. Evaluate $\lim_{x \to \infty} \left[\dfrac{1}{x} \cdot \dfrac{a^x - 1}{a - 1}\right]^{1/x}$ where $a > 0, a \neq 1$.

This problem was composed by the Committee on the Putnam Prize Competition.