8.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Determining Whether an Integral Is Improper In Exercises 1–8, decide whether the integral is improper. Explain your reasoning.

1. $\displaystyle\int_0^1 \frac{dx}{5x-3}$

2. $\displaystyle\int_1^2 \frac{dx}{x^3}$

3. $\displaystyle\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx$

4. $\displaystyle\int_1^\infty \ln(x^2)\,dx$

5. $\displaystyle\int_0^2 e^{-x}\,dx$

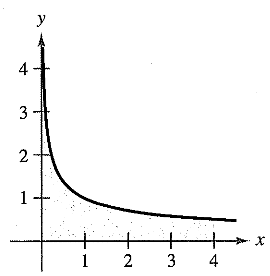
6. $\displaystyle\int_0^\infty \cos x\,dx$

7. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{4+x^2}\,dx$

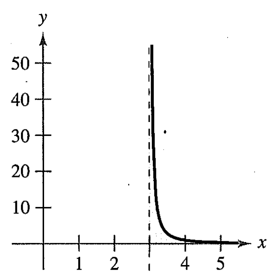
8. $\displaystyle\int_0^{\pi/4} \csc x\,dx$

Evaluating an Improper Integral In Exercises 9–12, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

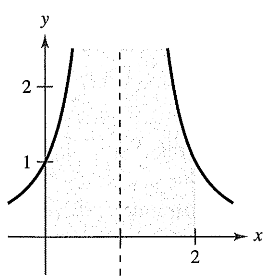
9. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



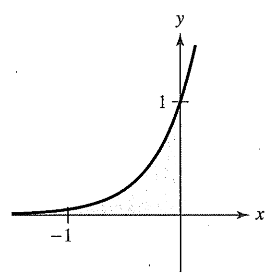
10. $\displaystyle\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx$



11. $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$



12. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$



Writing In Exercises 13–16, explain why the evaluation of the integral is *incorrect*. Use the integration capabilities of a graphing utility to attempt to evaluate the integral. Determine whether the utility gives the correct answer.

13. ~~$\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = -2$~~

14. ~~$\displaystyle\int_{-2}^2 \frac{-2}{(x-1)^3}\,dx = \frac{8}{9}$~~

15. ~~$\displaystyle\int_0^\infty e^{-x}\,dx = 0$~~

16. ~~$\displaystyle\int_0^\pi \sec x\,dx = 0$~~

Evaluating an Improper Integral In Exercises 17–32, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

17. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

18. $\displaystyle\int_1^\infty \frac{6}{x^4}\,dx$

19. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$

20. $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$

21. $\displaystyle\int_{-\infty}^0 xe^{-4x}\,dx$

22. $\displaystyle\int_0^\infty xe^{-x/3}\,dx$

23. $\displaystyle\int_0^\infty x^2e^{-x}\,dx$

24. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$

25. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$

26. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

27. $\displaystyle\int_{-\infty}^\infty \frac{4}{16+x^2}\,dx$

28. $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^2}\,dx$

29. $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$

30. $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}\,dx$

31. $\displaystyle\int_0^\infty \cos \pi x\,dx$

32. $\displaystyle\int_0^\infty \sin\frac{x}{2}\,dx$

Evaluating an Improper Integral In Exercises 33–48, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

33. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

34. $\displaystyle\int_0^5 \frac{10}{x}\,dx$

35. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

36. $\displaystyle\int_0^8 \frac{3}{\sqrt{8-x}}\,dx$

37. $\displaystyle\int_0^1 x\ln x\,dx$

38. $\displaystyle\int_0^e \ln x^2\,dx$

39. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

40. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$

41. $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx$

42. $\displaystyle\int_3^6 \frac{1}{\sqrt{36-x^2}}\,dx$

43. $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx$

44. $\displaystyle\int_0^5 \frac{1}{25-x^2}\,dx$

45. $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$

46. $\displaystyle\int_4^\infty \frac{\sqrt{x^2-16}}{x^2}\,dx$

47. $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$

48. $\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx$

Finding Values In Exercises 49 and 50, determine all values of p for which the improper integral converges.

49. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

50. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

51. Mathematical Induction Use mathematical induction to verify that the following integral converges for any positive integer n.

$$\int_0^{\infty} x^n e^{-x}\, dx$$

52. Comparison Test for Improper Integrals In some cases, it is impossible to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges. Suppose the functions f and g are continuous and $0 \le g(x) \le f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^{\infty} f(x)\, dx$ converges, then $\int_a^{\infty} g(x)\, dx$ also converges, and if $\int_a^{\infty} g(x)\, dx$ diverges, then $\int_a^{\infty} f(x)\, dx$ also diverges. This is known as the Comparison Test for improper integrals.

(a) Use the Comparison Test to determine whether $\int_1^{\infty} e^{-x^2}\, dx$ converges or diverges. (*Hint:* Use the fact that $e^{-x^2} \le e^{-x}$ for $x \ge 1$.)

(b) Use the Comparison Test to determine whether $\displaystyle\int_1^{\infty} \frac{1}{x^5 + 1}\, dx$ converges or diverges. (*Hint:* Use the fact that $\dfrac{1}{x^5 + 1} \le \dfrac{1}{x^5}$ for $x \ge 1$.)

Convergence or Divergence **In Exercises 53–62, use the results of Exercises 49–52 to determine whether the improper integral converges or diverges.**

53. $\displaystyle\int_0^1 \frac{1}{x^5}\, dx$

54. $\displaystyle\int_0^1 \frac{1}{\sqrt[5]{x}}\, dx$

55. $\displaystyle\int_1^{\infty} \frac{1}{x^5}\, dx$

56. $\displaystyle\int_0^{\infty} x^4 e^{-x}\, dx$

57. $\displaystyle\int_1^{\infty} \frac{1}{x^2 + 5}\, dx$

58. $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x - 1}}\, dx$

59. $\displaystyle\int_2^{\infty} \frac{1}{\sqrt[3]{x(x - 1)}}\, dx$

60. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}(x + 1)}\, dx$

61. $\displaystyle\int_1^{\infty} \frac{1 - \sin x}{x^2}\, dx$

62. $\displaystyle\int_0^{\infty} \frac{1}{e^x + x}\, dx$

WRITING ABOUT CONCEPTS

63. Improper Integrals Describe the different types of improper integrals.

64. Improper Integrals Define the terms *converges* and *diverges* when working with improper integrals.

65. Improper Integral Explain why $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\, dx \ne 0$.

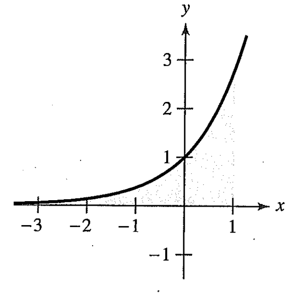
66. Improper Integral Consider the integral

$$\int_0^3 \frac{10}{x^2 - 2x}\, dx.$$

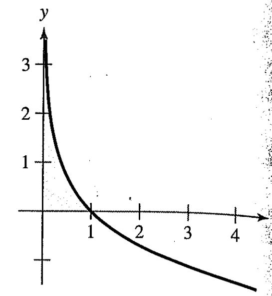
To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals if the given integral converges?

Area **In Exercises 67–70, find the area of the unbounded shaded region.**

67. $y = e^x, \quad -\infty < x \le 1$

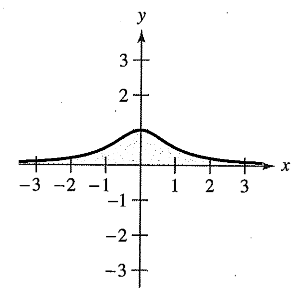


68. $y = -\ln x$



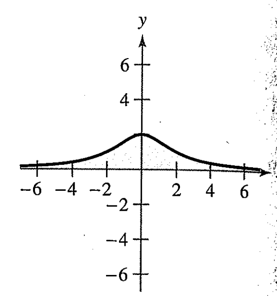
69. Witch of Agnesi:

$$y = \frac{1}{x^2 + 1}$$



70. Witch of Agnesi:

$$y = \frac{8}{x^2 + 4}$$



Area and Volume **In Exercises 71 and 72, consider the region satisfying the inequalities. (a) Find the area of the region. (b) Find the volume of the solid generated by revolving the region about the x-axis. (c) Find the volume of the solid generated by revolving the region about the y-axis.**

71. $y \le e^{-x},\ y \ge 0,\ x \ge 0$

72. $y \le \dfrac{1}{x^2},\ y \ge 0,\ x \ge 1$

73. Arc Length Sketch the graph of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$ and find its perimeter.

74. Arc Length Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

75. Surface Area The region bounded by $(x - 2)^2 + y^2 = 1$ is revolved about the y-axis to form a torus. Find the surface area of the torus.

76. Surface Area Find the area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, \infty)$ about the x-axis.

Propulsion **In Exercises 77 and 78, use the weight of the rocket to answer each question. (Use 4000 miles as the radius of Earth and do not consider the effect of air resistance.)**

(a) How much work is required to propel the rocket an unlimited distance away from Earth's surface?

(b) How far has the rocket traveled when half the total work has occurred?

77. 5-ton rocket

78. 10-ton rocket

Probability **A nonnegative function f is called a *probability density function* if**

$$\int_{-\infty}^{\infty} f(t)\,dt = 1.$$

The probability that x lies between a and b is given by

$$P(a \le x \le b) = \int_a^b f(t)\,dt.$$

The expected value of x is given by

$$E(x) = \int_{-\infty}^{\infty} t f(t)\,dt.$$

In Exercises 79 and 80, (a) show that the nonnegative function is a probability density function, (b) find $P(0 \le x \le 4)$, and (c) find $E(x)$.

79. $f(t) = \begin{cases} \frac{1}{7}e^{-t/7}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

80. $f(t) = \begin{cases} \frac{2}{5}e^{-2t/5}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

Capitalized Cost **In Exercises 81 and 82, find the capitalized cost C of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by**

$$C = C_0 + \int_0^n c(t)e^{-rt}\,dt$$

where C_0 is the original investment, t is the time in years, r is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance.

81. $C_0 = \$650{,}000$
$c(t) = \$25{,}000$
$r = 0.06$

82. $C_0 = \$650{,}000$
$c(t) = \$25{,}000(1 + 0.08t)$
$r = 0.06$

83. Electromagnetic Theory The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}}\,dx$$

where N, I, r, k, and c are constants. Find P.

84. Gravitational Force A "semi-infinite" uniform rod occupies the nonnegative x-axis. The rod has a linear density δ, which means that a segment of length dx has a mass of $\delta\,dx$. A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^{\infty} \frac{GM\delta}{(a + x)^2}\,dx$$

where G is the gravitational constant. Find F.

True or False? **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

86. If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) \ne 0$.

87. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then

$$\int_0^{\infty} f'(x)\,dx = -f(0).$$

88. If the graph of f is symmetric with respect to the origin or the y-axis, then $\int_0^{\infty} f(x)\,dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

89. Comparing Integrals

(a) Show that $\int_{-\infty}^{\infty} \sin x\,dx$ diverges.

(b) Show that $\lim_{a\to\infty} \int_{-a}^{a} \sin x\,dx = 0$.

(c) What do parts (a) and (b) show about the definition of improper integrals?

90. Making an Integral Improper For each integral, find a nonnegative real number b that makes the integral improper. Explain your reasoning.

(a) $\int_0^b \frac{1}{x^2 - 9}\,dx$ (b) $\int_0^b \frac{1}{\sqrt{4 - x}}\,dx$

(c) $\int_0^b \frac{x}{x^2 - 7x + 12}\,dx$ (d) $\int_b^{10} \ln x\,dx$

(e) $\int_0^b \tan 2x\,dx$ (f) $\int_0^b \frac{\cos x}{1 - \sin x}\,dx$

91. Writing

(a) The improper integrals

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx$$

diverge and converge, respectively. Describe the essential differences between the integrands that cause one integral to converge and the other to diverge.

(b) Sketch a graph of the function $y = (\sin x)/x$ over the interval $(1, \infty)$. Use your knowledge of the definite integral to make an inference as to whether the integral

$$\int_1^{\infty} \frac{\sin x}{x}\,dx$$

converges. Give reasons for your answer.

(c) Use one iteration of integration by parts on the integral in part (b) to determine its divergence or convergence.

92. Exploration Consider the integral

$$\int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$$

where n is a positive integer.

(a) Is the integral improper? Explain.

(b) Use a graphing utility to graph the integrand for $n = 2, 4, 8$, and 12.

(c) Use the graphs to approximate the integral as $n \to \infty$.

(d) Use a computer algebra system to evaluate the integral for the values of n in part (b). Make a conjecture about the value of the integral for any positive integer n. Compare your results with your answer in part (c).

93. Normal Probability The mean height of American men between 20 and 29 years old is 70 inches, and the standard deviation is 2.85 inches. A 20- to 29-year-old man is chosen at random from the population. The probability that he is 6 feet tall or taller is

$$P(72 \le x < \infty) = \int_{72}^{\infty} \frac{1}{2.85\sqrt{2\pi}} e^{-(x-70)^2/6.245}\, dx.$$

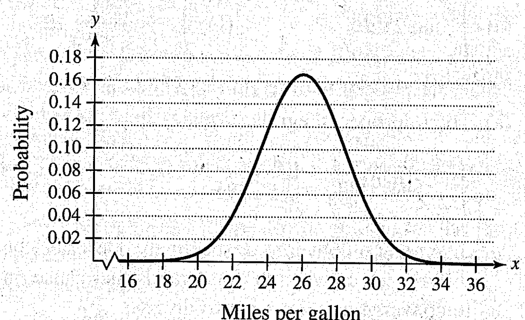
(*Source: National Center for Health Statistics*)

(a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the x-axis and the integrand is 1.

(b) Use a graphing utility to approximate $P(72 \le x < \infty)$.

(c) Approximate $0.5 - P(70 \le x \le 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

94. HOW DO YOU SEE IT? The graph shows the probability density function for a car brand that has a mean fuel efficiency of 26 miles per gallon and a standard deviation of 2.4 miles per gallon.



(a) Which is greater, the probability of choosing a car at random that gets between 26 and 28 miles per gallon or the probability of choosing a car at random that gets between 22 and 24 miles per gallon?

(b) Which is greater, the probability of choosing a car at random that gets between 20 and 22 miles per gallon or the probability of choosing a car at random that gets at least 30 miles per gallon?

Laplace Transforms Let $f(t)$ be a function defined for all positive values of t. The Laplace Transform of $f(t)$ is defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

when the improper integral exists. Laplace Transforms are used to solve differential equations. In Exercises 95–102, find the Laplace Transform of the function.

95. $f(t) = 1$

96. $f(t) = t$

97. $f(t) = t^2$

98. $f(t) = e^{at}$

99. $f(t) = \cos at$

100. $f(t) = \sin at$

101. $f(t) = \cosh at$

102. $f(t) = \sinh at$

103. The Gamma Function The Gamma Function $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx, \quad n > 0.$$

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n)$.

(c) Write $\Gamma(n)$ using factorial notation where n is a positive integer.

104. Proof Prove that $I_n = \left(\frac{n-1}{n+2}\right) I_{n-1}$, where

$$I_n = \int_0^{\infty} \frac{x^{2n-1}}{(x^2+1)^{n+3}}\, dx, \quad n \ge 1.$$

Then evaluate each integral.

(a) $\int_0^{\infty} \frac{x}{(x^2+1)^4}\, dx$

(b) $\int_0^{\infty} \frac{x^3}{(x^2+1)^5}\, dx$

(c) $\int_0^{\infty} \frac{x^5}{(x^2+1)^6}\, dx$

105. Finding a Value For what value of c does the integral

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx$$

converge? Evaluate the integral for this value of c.

106. Finding a Value For what value of c does the integral

$$\int_1^{\infty} \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx$$

converge? Evaluate the integral for this value of c.

107. Volume Find the volume of the solid generated by revolving the region bounded by the graph of f about the x-axis.

$$f(x) = \begin{cases} x \ln x, & 0 < x \le 2 \\ 0, & x = 0 \end{cases}$$

108. Volume Find the volume of the solid generated by revolving the unbounded region lying between $y = -\ln x$ and the y-axis $(y \ge 0)$ about the x-axis.

***u*-Substitution In Exercises 109 and 110, rewrite the improper integral as a proper integral using the given u-substitution. Then use the Trapezoidal Rule with $n = 5$ to approximate the integral.**

109. $\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx, \quad u = \sqrt{x}$

110. $\int_0^1 \frac{\cos x}{\sqrt{1-x}}\, dx, \quad u = \sqrt{1-x}$

111. Rewriting an Integral Let $\int_{-\infty}^{\infty} f(x)\, dx$ be convergent and let a and b be real numbers where $a \ne b$. Show that

$$\int_{-\infty}^{a} f(x)\, dx + \int_a^{\infty} f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx + \int_b^{\infty} f(x)\, dx.$$