59. $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$

From Example 5 you have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\sqrt{3}\right)^{2n}(2n+1)} \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n\left(1/\sqrt{3}\right)^{2n+1}}{2n+1}$$
$$= \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)$$
$$= \sqrt{3}\left(\frac{\pi}{6}\right) \approx 0.9068997$$

60. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n+1}}{(2n+1)!}$

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866025$$

61. Using a graphing utility, you obtain the following partial sums for the left hand side. Note that $1/\pi = 0.3183098862$.

$n = 0: S_0 \approx 0.3183098784$

$n = 1: S_1 = 0.3183098862$

62. You can verify that the statement is incorrect by calculating the constant terms of each side:

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = (1+1) + \left(x + \frac{x}{5}\right) + \cdots$$
$$\sum_{n=0}^{\infty}\left(1+\frac{1}{5}\right)x^n = \left(1+\frac{1}{5}\right) + \left(1+\frac{1}{5}\right)x + \cdots$$

The formula should be

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\left[1 + \left(\frac{1}{5}\right)^n\right]x^n.$$

## Section 9.10 Taylor and Maclaurin Series

1. For $c = 0$, you have:

$$f(x) = e^{2x}$$
$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$
$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

2. For $c = 0$, you have:

$$f(x) = e^{-4x}$$
$$f^{(n)}(x) = (-4)^n e^{-4x} \Rightarrow f^{(n)}(0) = (-4)^n$$
$$e^{-4x} = 1 - 4x + \frac{16x^2}{2!} - \frac{64x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^n}{n!}.$$

**3.** For $c = \pi/4$, you have:

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, you have:

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!}\\ &= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \cdots\right]\\ &= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^n}{n!}.\end{aligned}$$

[**Note:** $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \ldots$]

**4.** For $c = \pi/4$, you have:

$$f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore you have:

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!}\\ &= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots\right]\\ &= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}[x - (\pi/4)]^{n+1}}{(n+1)!} + 1\right\}.\end{aligned}$$

**5.** For $c = 1$, you have

$$f(x) = \frac{1}{x} = x^{-1} \qquad f(1) = 1$$
$$f'(x) = -x^{-2} \qquad f'(1) = -1$$
$$f''(x) = 2x^{-3} \qquad f''(1) = 2$$
$$f'''(x) = -6x^{-4} \qquad f'''(1) = -6$$

and so on. Therefore, you have

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$
$$= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots$$
$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

**6.** For $c = 2$, you have

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \qquad f(2) = -1$$
$$f'(x) = (1-x)^{-2} \qquad f'(2) = 1$$
$$f''(x) = 2(1-x)^{-3} \qquad f''(2) = -2$$
$$f'''(x) = 6(1-x)^{-4} \qquad f'''(2) = 6$$

and so on. Therefore you have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)(x-2)^n}{n!}$$
$$= -1 + (x-2) - (x-2)^2 + (x-2)^3 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$$

**7.** For $c = 1$, you have,

$$f(x) = \ln x \qquad f(1) = 0$$
$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$
$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$
$$f^{(4)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(1) = -6$$
$$f^{(5)}(x) = \frac{24}{x^5} \qquad f^{(5)}(1) = 24$$

and so on. Therefore, you have:

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$
$$= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \cdots$$
$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.$$

**8.** For $c = 1$, you have:

$$f(x) = e^x$$
$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e$$
$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$$

**9.** For $c = 0$, you have

$$\begin{aligned} f(x) &= \sin 3x & f(0) &= 0 \\ f'(x) &= 3\cos 3x & f'(0) &= 3 \\ f''(x) &= -9\sin 3x & f''(0) &= 0 \\ f'''(x) &= -27\cos 3x & f'''(0) &= -27 \\ f^{(4)}(x) &= 81\sin 3x & f^{(4)}(0) &= 0 \end{aligned}$$

and so on. Therefore you have

$$\sin 3x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 3x + 0 - \frac{27x^3}{3!} + 0 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

**10.** For $c = 0$, you have.

$$\begin{aligned} f(x) &= \ln(x^2 + 1) & f(0) &= 0 \\ f'(x) &= \frac{2x}{x^2 + 1} & f'(0) &= 0 \\ f''(x) &= \frac{2 - 2x^2}{(x^2 + 1)^2} & f''(0) &= 2 \\ f'''(x) &= \frac{4x(x^2 - 3)}{(x^2 + 1)^3} & f'''(0) &= 0 \\ f^{(4)}(x) &= \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} & f^{(4)}(0) &= -12 \\ f^{(5)}(x) &= \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} & f^{(6)}(0) &= 240 \end{aligned}$$

and so on. Therefore, you have:

$$\ln(x^2 + 1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}.$$

**11.** For $c = 0$, you have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5\sec^5(x) + 18\sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \end{aligned}$$

$$\sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots.$$

**12.** For $c = 0$, you have;

$$f(x) = \tan(x) \qquad f(0) = 0$$
$$f'(x) = \sec^2(x) \qquad f'(0) = 1$$
$$f''(x) = 2\sec^2(x)\tan(x) \qquad f''(0) = 0$$
$$f'''(x) = 2\left[\sec^4(x) + 2\sec^2(x)\tan^2(x)\right] \qquad f'''(0) = 2$$
$$f^{(4)}(x) = 8\left[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)\right] \qquad f^{(4)}(0) = 0$$
$$f^{(5)}(x) = 8\left[2\sec^6(x) + 11\sec^4(x)\tan^2(x) + 2\sec^2(x)\tan^4(x)\right] \qquad f^{(5)}(0) = 16$$
$$\tan(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots.$$

**13.** The Maclaurin series for $f(x) = \cos x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, you have $\left|f^{(n+1)}(z)\right| \le 1$ for all $z$. So by Taylor's Theorem,

$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Because $\displaystyle\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$. So, the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x$.

**14.** The Maclaurin series for $f(x) = e^{-2x}$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. So, by Taylor's Theorem,

$$0 \le \left|Rn(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{(-2)^{n+1}e^{-2z}}{(n+1)!}x^{n+1}\right|.$$

Because $\displaystyle\lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)!}\right| = 0$, it follows that $R_n(x) \to 0$ as $n \to \infty$.

So, the Maclaurin Series for $e^{-2x}$ converges to $e^{-2x}$ for all $x$.

**15.** The Maclaurin series for $f(x) = \sinh x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed $x$,

$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$ ). So, the Maclaurin series for $\sinh x$ converges to $\sinh x$ for all $x$.

**16.** The Maclaurin series for $f(x) = \cosh x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

$f^{(n+1)}(x) = \sinh x$ (or $\cosh x$). For fixed $x$,

$$0 \le |R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| = \left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right| \to 0 \text{ as } n \to \infty.$$

(The argument is the same if $f^{(n+1)}(x) = \cosh x$). So, the Maclaurin series for $\cosh x$ converges to $\cosh x$ for all $x$.

**17.** Because $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, you have

$$(1+x)^{-2} = 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{4!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n.$$

**18.** Because $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$

you have

$$(1+x)^{-4} = 1 - 4x + \frac{4(5)}{2!}x^2 - \frac{4(5)(6)}{3!}x^3 + \frac{4(5)(6)(7)}{4!}x^4 = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(n+3)!}{3!n!}x^n$$

**19.** Because $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, you have

$$[1+(-x)]^{-1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}.$$

**20.** Because $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$ you have

$$\left[1+(-x^2)\right]^{-1/2} = 1 - \frac{1}{2}x^2 + \frac{(1/2)(3/2)}{2!}x^4 - \frac{(1/2)(3/2)(5/2)x^6}{3!} + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{(1)(3)}{2^2 2!}x^4 - \frac{(1)(3)(5)}{2^3 3!}x^6 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}x^{2n}$$

**21.** $\dfrac{1}{\sqrt{4+x^2}} = \left(\dfrac{1}{2}\right)\left[1 + \left(\dfrac{x}{2}\right)^2\right]^{-1/2}$ and because $(1+x)^{-1/2} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, you have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

**22.** $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\left(1 + \dfrac{x}{2}\right)^{-3}, \qquad k = -3$

$$\frac{1}{(2+x)^3} = \frac{1}{8}\left\{1 - 3\left(\frac{x}{2}\right) + \frac{3(4)}{2!}\left(\frac{x}{2}\right)^2 - \frac{3(4)(5)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right\} = \frac{1}{8}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(n+2)!}{2^{n+1}n!}x^n\right]$$

**23.** $\sqrt{1+x} = (1+x)^{1/2}, \qquad k = 1/2$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1/2(-1/2)}{2!}x^2 + \frac{1/2(-1/2)(-3/2)}{3!}x^3 + \cdots = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^n$$

**24.** 
$$(1+x)^{1/4} = 1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \frac{(1/4)(-3/4)(-7/4)}{3!}x^3 + \cdots$$
$$= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3\cdot 7}{4^3 3!}x^3 - \frac{3\cdot 7\cdot 11}{4^4 4!}x^4 + \cdots$$
$$= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!}x^n$$

**25.** Because $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}$

you have $(1+x^2)^{1/2} = 1 + \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^{2n}}{2^n n!}$.

**26.** Because $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}$

you have $(1+x^3)^{1/2} = 1 + \dfrac{x^3}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^{3n}}{2^n n!}$.

**27.**
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$
$$e^{x^2/2} = \sum_{n=0}^{\infty}\frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

**28.**
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$
$$e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

**29.**
$$\ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}, \; 0 < x \le 2$$
$$\ln(x+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, \; -1 < x \le 1$$

**30.**
$$\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}, \; 0 < x \le 2$$
$$\ln(x^2+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}, \; -1 < x \le 1$$

**31.**
$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\sin 3x = \sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{(2n+1)!}$$

**32.**
$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\sin \pi x = \sum_{n=0}^{\infty}\frac{(-1)^n(\pi x)^{2n+1}}{(2n+1)!}$$

**33.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$
$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

**34.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\cos \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

**35.** $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$$
$$= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

**36.** $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$2 \sin x^3 = 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$
$$= 2\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right)$$
$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

**37.** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$
$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$
$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

**38.** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$
$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$
$$2 \cos h(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$$

**39.** $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \cdots\right] = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$$

**40.** The formula for the binomial series gives $(1 + x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, which implies that

$$(1 + x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!}$$
$$\ln\left(x + \sqrt{x^2 + 1}\right) = \int \frac{1}{\sqrt{x^2 + 1}}\, dx$$
$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2^n (2n+1) n!}$$
$$= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$$

**41.** $x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

**42.** $x\cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$

**43.** $\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$

$= 1, x = 0$

**44.** $\frac{\arcsin x}{x} = \sum_{n=0}^{\infty}\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}\cdot\frac{1}{x} = \sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{(2^n n!)^2(2n+1)}, x \neq 0$

$= 1, x = 0$

**45.**

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

**46.** $e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots$ (See Exercise 45.)

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

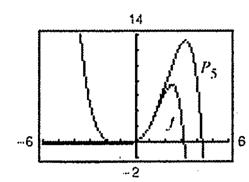
**47.** $f(x) = e^x \sin x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

$$P_5(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$$



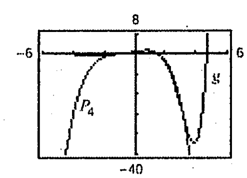

**48.** $g(x) = e^x \cos x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

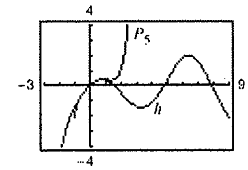
$$= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

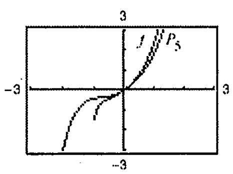
$$P_4(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$



**49.**
$$\begin{aligned} h(x) &= \cos x \ln(1+x) \\ &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots \\ &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \\ P_5(x) &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} \end{aligned}$$

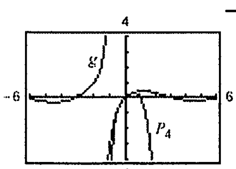



**50.**
$$\begin{aligned} f(x) &= e^x \ln(1+x) \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \\ P_5(x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} \end{aligned}$$




**51.** $g(x) = \dfrac{\sin x}{1+x}$. Divide the series for $\sin x$ by $(1 + x)$.

$$\begin{array}{r|l} & x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \\ \hline 1 + x & x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots \\ & \underline{x + x^2} \\ & -x^2 - \frac{x^3}{6} \\ & \underline{-x^2 - x^3} \\ & \frac{5x^3}{6} + 0x^4 \\ & \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\ & -\frac{5x^4}{6} + \frac{x^5}{120} \\ & \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\ & \vdots \end{array}$$

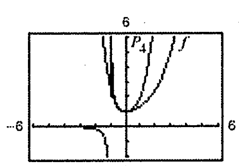



$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$

$$P_4(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6}$$

**52.** $f(x) = \dfrac{e^x}{1+x}$. Divide the series for $e^x$ by $(1 + x)$.

$$\begin{array}{r|l} & 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots \\ \hline 1 + x & 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \\ & \underline{1 + x} \\ & 0 + \frac{x^2}{2} + \frac{x^3}{6} \\ & \underline{\frac{x^2}{2} + \frac{x^3}{2}} \\ & -\frac{x^3}{3} + \frac{x^4}{24} \\ & \underline{-\frac{x^3}{3} - \frac{x^4}{3}} \\ & \frac{3x^4}{8} + \frac{x^5}{120} \\ & \underline{\frac{3x^4}{8} + \frac{3x^5}{8}} \\ & \vdots \end{array}$$




$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$

$$P_4(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8}$$

**53.** $$\int_0^x \left(e^{-t^2} - 1\right) dt = \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) - 1\right] dt$$
$$= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!}\right] dt$$
$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!}\right]_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}$$

**54.** $$\int_0^x \sqrt{1+t^3}\, dt = \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!}\right] dt$$
$$= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1) 2^n n!}\right]_0^x$$
$$= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1) 2^n n!}$$

**55.** Because $\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad (0 < x \le 2)$

you have $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931.$ (10,001 terms)

**56.** Because $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, you have

$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415.$ (4 terms)

**57.** Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

you have $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891.$ (12 terms)

**58.** Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, you have $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

and $\frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321.$ (6 terms)

**59.** Because

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

you have $\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+2)!} = 0.$

**60.** Because

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

you have $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$

**61.** Because $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

and $\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$

you have $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \sum \frac{x^n}{(n+1)!} = 1.$

**62.** Because $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(See Exercise 29.)

$$\frac{\ln(x+1)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$$

you have $\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = 1.$

**63.**
$$\int_0^1 e^{-x^3}\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!}\right] dx$$
$$= \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}\right] dx$$
$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)n!}\right]_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)n!}$$
$$= 1 - \frac{1}{4} + \frac{1}{14} - \cdots + (-1)^n \frac{1}{(3n+1)n!} + \cdots$$

Because $\frac{1}{[3(6)+1]6!} < 0.0001$, you need 6 terms.

$$\int_0^1 e^{-x^2}\,dx \approx \sum_{n=0}^{5} \frac{(-1)^n}{(3n+1)n!} \approx 0.8075$$

**64.** $\int_0^{1/4} x\ln(x+1)\,dx = \int_0^{1/4}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)dx = \left[\frac{x^3}{3} - \frac{x^4}{4\cdot 2} + \frac{x^5}{5\cdot 3} - \frac{x^6}{6\cdot 4} + \cdots\right]_0^{1/4}$

Because $\frac{(1/4)^5}{15} < 0.0001$, $\int_0^{1/4} x\ln(x+1)\,dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$

**65.** $\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\right]dx = \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}$

Because $1/(7\cdot 7!) < 0.0001$, you need three terms:

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \cdots \approx 0.9461. \quad \text{(using three nonzero terms)}$$

**Note:** You are using $\lim_{x\to 0^+}\frac{\sin x}{x} = 1$.

**66.** $\int_0^1 \cos x^2\,dx = \int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}\right]dx$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)(2n)!}$$

$$\int_0^1 \cos x^2\,dx \approx \sum_{n=0}^{3}\frac{(-1)^n}{(4n+1)(2n)!} \approx 0.904523$$

Because $\frac{1}{[4(4)+1][2(4)]!} < 0.0001$, you need 4 terms.

**67.** $\int_0^{1/2}\frac{\arctan x}{x}\,dx = \int_0^{1/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right)dx$

$$= \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$$

Because $1/(9^2 2^9) < 0.0001$, you have

$$\int_0^{1/2}\frac{\arctan x}{x}\,dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

**Note:** You are using $\lim_{x\to 0^+}\frac{\arctan x}{x} = 1$.

**68.** $\int_0^{1/2}\arctan(x^2)\,dx = \int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2n+1}\right]dx$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)(2n+1)2^{4n+3}}$$

Because $\frac{1}{(4n+3)(2n+1)2^{4n+3}} < 0.0001$

when $n = 2$, you need 2 terms.

$$\int_0^{1/2}\arctan(x^2)\,dx \approx \frac{1}{3(1)\cdot 2^3} - \frac{1}{7(3)2^7} \approx 0.041295$$

**69.** $\int_{0.1}^{0.3}\sqrt{1+x^3}\,dx = \int_{0.1}^{0.3}\left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots\right)dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots\right]_{0.1}^{0.3}$

Because $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, you need two terms.

$$\int_{0.1}^{0.3}\sqrt{1+x^3}\,dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4)\right] \approx 0.201.$$

**70.** $\sqrt{1+x^2} = (1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^4}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^6}{3!} + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots$

$$\int_0^{0.2} \sqrt{1+x^2}\,dx = \int_0^{0.2}\left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots\right]dx = \left[x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} - \cdots\right]_0^{0.2}$$

Because $\frac{(0.2)^5}{40} < 0.0001$, you need 2 terms.

$$\int_0^{0.2} \sqrt{1+x^2}\,dx \approx 0.2 + \frac{(0.2)^3}{6} \approx 0.201333$$

**71.** $\int_0^{\pi/2} \sqrt{x}\cos x\,dx = \int_0^{\pi/2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!}\right]dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!}\right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!}\right]_0^{\pi/2}$

Because $2(\pi/2)^{23/2}/(23 \cdot 10!) < 0.0001$, you need five terms.

$$\int_0^1 \sqrt{x}\cos x\,dx = 2\left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10{,}800} + \frac{(\pi/2)^{19/2}}{766{,}080}\right] \approx 0.7040.$$

**72.** $\int_{0.5}^1 \cos\sqrt{x}\,dx = \int_{0.5}^1\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right)dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$

Because $\frac{1}{201{,}600}\left(1 - 0.5^5\right) < 0.0001$, you have

$$\int_{0.5}^1 \cos\sqrt{x}\,dx \approx \left[(1-0.5) - \frac{1}{4}\left(1-0.5^2\right) + \frac{1}{72}\left(1-0.5^3\right) - \frac{1}{2880}\left(1-0.5^4\right) + \frac{1}{201{,}600}\left(1-0.5^5\right)\right] \approx 0.3243.$$

**73.** From Exercise 27, you have

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_0^1 = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{2\cdot 1\cdot 3} + \frac{1}{2^2\cdot 2!\cdot 5} - \frac{1}{2^3\cdot 3!\cdot 7}\right) \approx 0.3412.$$

**74.** From Exercise 27, you have

$$\frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_1^2 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_1^2$$

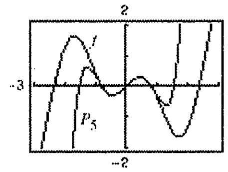
$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(2^{n+1}-1\right)}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left(1 - \frac{7}{2\cdot 1\cdot 3} + \frac{31}{2^2\cdot 2!\cdot 5} - \frac{127}{2^3\cdot 3!\cdot 7} + \frac{511}{2^4\cdot 4!\cdot 9} - \frac{2047}{2^5\cdot 5!\cdot 11}\right.$$

$$\left. + \frac{8191}{2^6\cdot 6!\cdot 13} - \frac{32{,}767}{2^7\cdot 7!\cdot 15} + \frac{131{,}071}{2^8\cdot 8!\cdot 17} - \frac{524{,}287}{2^9\cdot 9!\cdot 19}\right) \approx 0.1359.$$

**75.** $f(x) = x\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

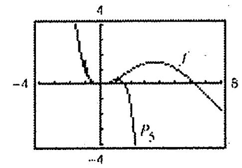
$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$




The polynomial is a reasonable approximation on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

**76.** $f(x) = \sin\frac{x}{2}\ln(1 + x)$

$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$

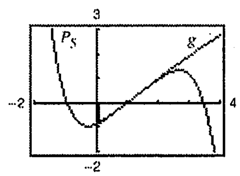



The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.

**77.** $f(x) = \sqrt{x}\ln x, c = 1$

$P_5(x) = (x - 1) - \frac{(x - 1)^3}{24} + \frac{(x - 1)^4}{24} - \frac{71(x - 1)^5}{1920}$

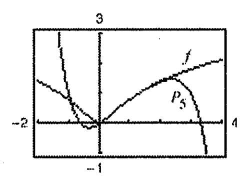
The polynomial is a reasonable approximation on the interval $\left[\frac{1}{4}, 2\right]$.




**78.** $f(x) = \sqrt[3]{x} \cdot \arctan x, c = 1$

$P_5(x) \approx 0.7854 + 0.7618(x - 1) - 0.3412\left[\frac{(x - 1)^2}{2!}\right] - 0.0424\left[\frac{(x - 1)^3}{3!}\right] + 1.3025\left[\frac{(x - 1)^4}{4!}\right] - 5.5913\left[\frac{(x - 1)^5}{5!}\right]$

The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.




**79.** See Guidelines, page 668.

**80.** The binomial series is $(1 + x)^k = 1 + kx + \frac{k(k - 1)}{2!}x^2 + \frac{k(k - 1)(k - 2)}{3!}x^3 + \cdots$. The radius of convergence is $R = 1$.

**81.** (a) Replace $x$ with $(-x)$.

(b) Replace $x$ with $3x$.

(c) Multiply series by $x$.

(d) Replace $x$ with $2x$, then replace $x$ with $-2x$, and add the two together.

**82.** (a) $y = x^2 - \frac{x^4}{3!} \Rightarrow$ even polynomial, degree 4

Matches (iii).

$$y = x\left(x - \frac{x^3}{3!}\right)$$

The second factor is the third-degree Taylor polynomial for $f(x) = \sin x$ at $c = 0$.

(b) $y = x - \frac{x^3}{2!} + \frac{x^5}{4!} \Rightarrow$ odd polynomial, degree 5

Matches (iv).

$$y = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$

The second factor is the fourth-degree Taylor polynomial for $f(x) = \cos x$ at $c = 0$.

(c) $y = x + x^2 + \frac{x^3}{2!} \Rightarrow$ odd polynomial, degree 3

Matches (i).

$$y = x\left(1 + x + \frac{x^2}{2!}\right)$$

The second factor is the third-degree Taylor polynomial for $f(x) = e^x$ at $c = 0$.

(d) $y = x^2 - x^3 + x^4 \Rightarrow$ even polynomial, degree 4

Matches (ii).

$$y = x^2(1 - x + x^2)$$

The second factor is the second-degree Taylor polynomial for $f(x) = \frac{1}{1 + x}$ at $c = 0$.

**83.** $$y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$$

$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} - \frac{g}{k^2}\left[-\frac{kx}{v_0\cos\theta} - \frac{1}{2}\left(\frac{kx}{v_0\cos\theta}\right)^2 - \frac{1}{3}\left(\frac{kx}{v_0\cos\theta}\right)^3 - \frac{1}{4}\left(\frac{kx}{v_0\cos\theta}\right)^4 - \cdots\right]$$

$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} + \frac{gx}{kv_0\cos\theta} + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{gkx^3}{3v_0^3\cos^3\theta} + \frac{gk^2x^4}{4v_0^4\cos^4\theta} + \cdots$$

$$= (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots$$

**84.** $\theta = 60°, v_0 = 64, k = \frac{1}{16}, g = -32$

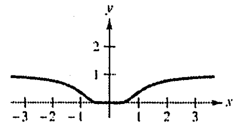
$$y = \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots$$

$$= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]$$

$$= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^nx^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}$$

**85.** $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x \to 0} \dfrac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x \to 0} \ln\left(\frac{e^{-1/x^2}}{x}\right) = \lim_{x \to 0^+}\left[-\frac{1}{x^2} - \ln x\right] = \lim_{x \to 0^+}\left[\frac{-1 - x^2 \ln x}{x^2}\right] = -\infty.$$

So, $y = e^{-\infty} = 0$ and you have $f'(0) = 0$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = 0 \neq f(x)$ This series converges to $f$ at $x = 0$ only.

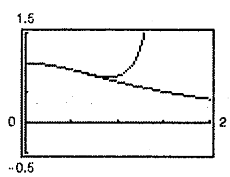
**86.** (a) $f(x) = \dfrac{\ln(x^2 + 1)}{x^2}$.

From Exercise 10, you obtain:

$$P = \frac{1}{x^2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n + 1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}.$$

(b)



(c) $F(x) = \displaystyle\int_0^x \frac{\ln(t^2 + 1)}{t^2}\,dt$

$G(x) = \displaystyle\int_0^x P_8(t)\,dt$

| $x$ | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 |
|---|---|---|---|---|---|---|
| $F(x)$ | 0.2475 | 0.4810 | 0.6920 | 0.8776 | 1.1798 | 1.4096 |
| $G(x)$ | 0.2475 | 0.4810 | 0.6924 | 0.8865 | 1.6878 | 9.6063 |

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

**87.** By the Ratio Test: $\displaystyle\lim_{n \to \infty}\left|\frac{x^{n+1}}{(n + 1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n \to \infty} \frac{|x|}{n + 1} = 0$ which shows that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x$.

**88.** $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots) = 2x\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1},\ R = 1$$

$$\ln 3 = \ln\left(\frac{1+1/2}{1-1/2}\right) \approx 2\left(\frac{1}{2}\right)\left[1 + \frac{(1/2)^2}{3} + \frac{(1/2)^4}{5} + \frac{(1/2)^6}{7}\right] = 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} \approx 1.098065$$

$(\ln 3 \approx 1.098612)$

**89.** $\binom{5}{3} = \frac{5\cdot 4\cdot 3}{3!} = \frac{60}{6} = 10$

**90.** $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$

**91.** $\binom{0.5}{4} = \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!} = -0.0390625 = -\frac{5}{128}$

**92.** $\binom{-1/3}{5} = \frac{(-1/3)(-4/3)(-7/3)(-10/3)(-13/3)}{5!}$

$= \frac{-91}{729} \approx -0.12483$

**93.** $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$

Example: $(1+x)^2 = \sum_{n=0}^{\infty}\binom{2}{n}x^n = 1 + 2x + x^2$

**94.** Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \cdots + \frac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1 - \left(\frac{1}{N+1}\right)}\right] = \frac{1}{N},$$ a contradiction.

**95.** $g(x) = \frac{x}{1 - x - x^2} = a_0 + a_1x + a_2x^2 + \cdots$

$x = (1 - x - x^2)(a_0 + a_1x + a_2x^2 + \cdots)$

$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots$

Equating coefficients,

$a_0 = 0$

$a_1 - a_0 = 1 \Rightarrow a_1 = 1$

$a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = 1$

$a_3 - a_2 - a_1 = 0 \Rightarrow a_3 = 2$

$a_4 = a_3 + a_2 = 3$, etc.

In general, $a_n = a_{n-1} + a_{n-2}$. The coefficients are the Fibonacci numbers.