9.10 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Taylor Series In Exercises 1–12, use the definition of Taylor series to find the Taylor series, centered at c, for the function.

1. $f(x) = e^{2x}, \quad c = 0$
2. $f(x) = e^{-4x}, \quad c = 0$
3. $f(x) = \cos x, \quad c = \frac{\pi}{4}$
4. $f(x) = \sin x, \quad c = \frac{\pi}{4}$
5. $f(x) = \frac{1}{x}, \quad c = 1$
6. $f(x) = \frac{1}{1 - x}, \quad c = 2$
7. $f(x) = \ln x, \quad c = 1$
8. $f(x) = e^x, \quad c = 1$
9. $f(x) = \sin 3x, \quad c = 0$
10. $f(x) = \ln(x^2 + 1), \quad c = 0$
11. $f(x) = \sec x, \quad c = 0$ (first three nonzero terms)
12. $f(x) = \tan x, \quad c = 0$ (first three nonzero terms)

Proof In Exercises 13–16, prove that the Maclaurin series for the function converges to the function for all x.

13. $f(x) = \cos x$
14. $f(x) = e^{-2x}$
15. $f(x) = \sinh x$
16. $f(x) = \cosh x$

Using a Binomial Series In Exercises 17–26, use the binomial series to find the Maclaurin series for the function.

17. $f(x) = \frac{1}{(1 + x)^2}$
18. $f(x) = \frac{1}{(1 + x)^4}$
19. $f(x) = \frac{1}{\sqrt{1 - x}}$
20. $f(x) = \frac{1}{\sqrt{1 - x^2}}$
21. $f(x) = \frac{1}{\sqrt{4 + x^2}}$
22. $f(x) = \frac{1}{(2 + x)^3}$
23. $f(x) = \sqrt{1 + x}$
24. $f(x) = \sqrt[4]{1 + x}$
25. $f(x) = \sqrt{1 + x^2}$
26. $f(x) = \sqrt{1 + x^3}$

Finding a Maclaurin Series In Exercises 27–40, find the Maclaurin series for the function. Use the table of power series for elementary functions on page 670.

27. $f(x) = e^{x^2/2}$
28. $g(x) = e^{-3x}$
29. $f(x) = \ln(1 + x)$
30. $f(x) = \ln(1 + x^2)$
31. $g(x) = \sin 3x$
32. $f(x) = \sin \pi x$
33. $f(x) = \cos 4x$
34. $f(x) = \cos \pi x$
35. $f(x) = \cos x^{3/2}$
36. $g(x) = 2 \sin x^3$
37. $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$
38. $f(x) = e^x + e^{-x} = 2 \cosh x$
39. $f(x) = \cos^2 x$
40. $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\left(\textit{Hint:} \text{ Integrate the series for } \frac{1}{\sqrt{x^2 + 1}}.\right)$

Finding a Maclaurin Series In Exercises 41–44, find the Maclaurin series for the function. (See Examples 7 and 8.)

41. $f(x) = x \sin x$
42. $h(x) = x \cos x$
43. $g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$
44. $f(x) = \begin{cases} \frac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Verifying a Formula In Exercises 45 and 46, use a power series and the fact that $i^2 = -1$ to verify the formula.

45. $g(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$
46. $g(x) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

Finding Terms of a Maclaurin Series In Exercises 47–52, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 670. Use a graphing utility to graph the function and its corresponding polynomial approximation.

47. $f(x) = e^x \sin x$
48. $g(x) = e^x \cos x$
49. $h(x) = \cos x \ln(1 + x)$
50. $f(x) = e^x \ln(1 + x)$
51. $g(x) = \frac{\sin x}{1 + x}$
52. $f(x) = \frac{e^x}{1 + x}$

Finding a Maclaurin Series In Exercises 53 and 54, find a Maclaurin series for $f(x)$.

53. $f(x) = \int_0^x (e^{-t^2} - 1)\, dt$
54. $f(x) = \int_0^x \sqrt{1 + t^3}\, dt$

Verifying a Sum In Exercises 55–58, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.

55. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$
56. $\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n + 1)!}\right] = \sin 1$
57. $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$
58. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n!}\right) = \frac{e - 1}{e}$

Finding a Limit In Exercises 59–62, use the series representation of the function f to find $\lim_{x \to 0} f(x)$ (if it exists).

59. $f(x) = \frac{1 - \cos x}{x}$
60. $f(x) = \frac{\sin x}{x}$
61. $f(x) = \frac{e^x - 1}{x}$
62. $f(x) = \frac{\ln(x + 1)}{x}$

Approximating an Integral **In Exercises 63–70, use a power series to approximate the value of the integral with an error of less than 0.0001. (In Exercises 65 and 67, assume that the integrand is defined as 1 when $x = 0$.)**

63. $\int_0^1 e^{-x^3}\, dx$

64. $\int_0^{1/4} x \ln(x+1)\, dx$

65. $\int_0^1 \frac{\sin x}{x}\, dx$

66. $\int_0^1 \cos x^2\, dx$

67. $\int_0^{1/2} \frac{\arctan x}{x}\, dx$

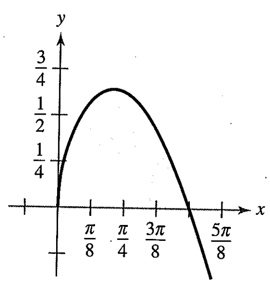
68. $\int_0^{1/2} \arctan x^2\, dx$

69. $\int_{0.1}^{0.3} \sqrt{1+x^3}\, dx$

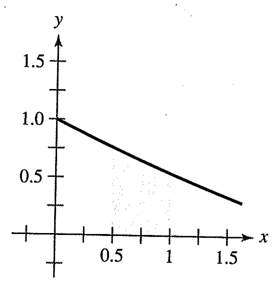
70. $\int_0^{0.2} \sqrt{1+x^2}\, dx$

Area **In Exercises 71 and 72, use a power series to approximate the area of the region. Use a graphing utility to verify the result.**

71. $\int_0^{\pi/2} \sqrt{x}\cos x\, dx$

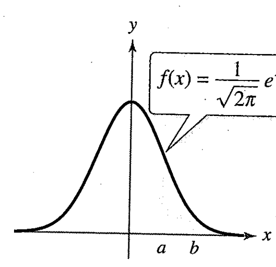


72. $\int_{0.5}^{1} \cos\sqrt{x}\, dx$



Probability **In Exercises 73 and 74, approximate the normal probability with an error of less than 0.0001, where the probability is given by**

$$P(a < x < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx.$$



73. $P(0 < x < 1)$ **74.** $P(1 < x < 2)$

Finding a Taylor Polynomial Using Technology **In Exercises 75–78, use a computer algebra system to find the fifth-degree Taylor polynomial, centered at c, for the function. Graph the function and the polynomial. Use the graph to determine the largest interval on which the polynomial is a reasonable approximation of the function.**

75. $f(x) = x\cos 2x, \quad c = 0$

76. $f(x) = \sin\frac{x}{2}\ln(1+x), \quad c = 0$

77. $g(x) = \sqrt{x}\ln x, \quad c = 1$

78. $h(x) = \sqrt[3]{x}\arctan x, \quad c = 1$

WRITING ABOUT CONCEPTS

79. Taylor Series State the guidelines for finding a Taylor series.

80. Binomial Series Define the binomial series. What is its radius of convergence?

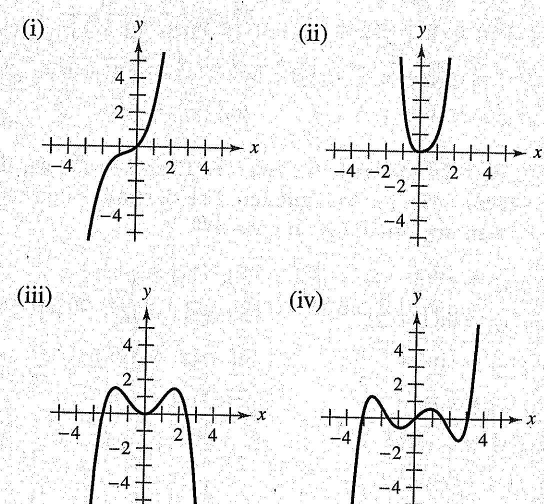
81. Finding a Series Explain how to use the series

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to find the series for each function. Do not find the series.

(a) $f(x) = e^{-x}$ (b) $f(x) = e^{3x}$ (c) $f(x) = xe^x$

82. HOW DO YOU SEE IT? Match the polynomial with its graph. [The graphs are labeled (i), (ii), (iii), and (iv).] Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.



(a) $y = x^2 - \frac{x^4}{3!}$ (b) $y = x - \frac{x^3}{2!} + \frac{x^5}{4!}$

(c) $y = x + x^2 + \frac{x^3}{2!}$ (d) $y = x^2 - x^3 + x^4$

83. Projectile Motion A projectile fired from the ground follows the trajectory given by

$$y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$$

where v_0 is the initial speed, θ is the angle of projection, g is the acceleration due to gravity, and k is the drag factor caused by air resistance. Using the power series representation

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1$$

verify that the trajectory can be rewritten as

$$y = (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots.$$

84. Projectile Motion

Use the result of Exercise 83 to determine the series for the path of a projectile launched from ground level at an angle of $\theta = 60°$, with an initial speed of $v_0 = 64$ feet per second and a drag factor of $k = \frac{1}{16}$.

85. Investigation Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Sketch a graph of the function.

(b) Use the alternative form of the definition of the derivative (Section 2.1) and L'Hôpital's Rule to show that $f'(0) = 0$. [By continuing this process, it can be shown that $f^{(n)}(0) = 0$ for $n > 1$.]

(c) Using the result in part (b), find the Maclaurin series for f. Does the series converge to f?

86. Investigation

(a) Find the power series centered at 0 for the function

$$f(x) = \frac{\ln(x^2+1)}{x^2}.$$

(b) Use a graphing utility to graph f and the eighth-degree Taylor polynomial $P_8(x)$ for f.

(c) Complete the table, where

$$F(x) = \int_0^x \frac{\ln(t^2+1)}{t^2}\,dt \quad \text{and} \quad G(x) = \int_0^x P_8(t)\,dt.$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$						
$G(x)$						

(d) Describe the relationship between the graphs of f and P_8 and the results given in the table in part (c).

87. Proof Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real x.

88. Finding a Maclaurin Series Find the Maclaurin series for

$$f(x) = \ln\frac{1+x}{1-x}$$

and determine its radius of convergence. Use the first four terms of the series to approximate ln 3.

Evaluating a Binomial Coefficient **In Exercises 89–92, evaluate the binomial coefficient using the formula**

$$\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-n+1)}{n!}$$

where k is a real number, n is a positive integer, and

$$\binom{k}{0} = 1.$$

89. $\binom{5}{3}$

90. $\binom{-2}{2}$

91. $\binom{0.5}{4}$

92. $\binom{-1/3}{5}$

93. Writing a Power Series Write the power series for $(1+x)^k$ in terms of binomial coefficients.

94. Proof Prove that e is irrational. [*Hint:* Assume that $e = p/q$ is rational (p and q are integers) and consider

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots.]$$

95. Using Fibonacci Numbers Show that the Maclaurin series for the function

$$g(x) = \frac{x}{1-x-x^2}$$

is

$$\sum_{n=1}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number with $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$, for $n \geq 3$.

(*Hint:* Write

$$\frac{x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \cdots$$

and multiply each side of this equation by $1 - x - x^2$.)

PUTNAM EXAM CHALLENGE

96. Assume that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all x on an interval of length at least 2. Show that $|f'(x)| \leq 2$ on the interval.