Key

BC Calculus – 9.1 Notes – Defining and Differentiating Parametric Equations

We have been looking at graphs of one equation with two variables, typically x and y. Now we are looking at three variables that will represent a curve in the plane.

→ t variable is the parameter (usually deals with time)

In the rectangular equation we are able to determine where the object is located at a point (x, y), but with the addition of the third variable (often t), we are able to determine when the object was at a point (x, y). NOTE: the third variable t is often time, but not always.

Parametric Equations

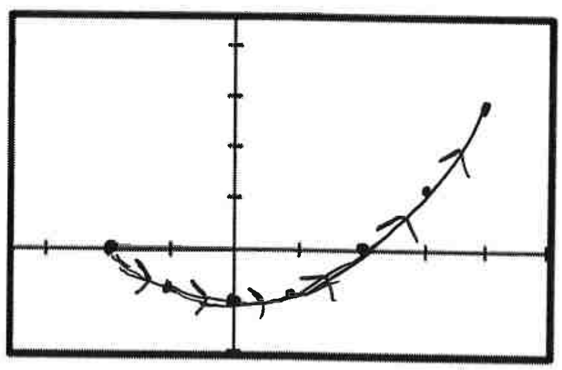
If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are parametric equations and t is the parameter. You can sketch the curve of a parametric by substituting in values for t.

1. Sketch the curve with the following parametrization: $x(t) = 2t$ and $y(t) = t^2 - 1$, with $-1 \le t \le 2$.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	-2	-1	0	1	2	3	4
y	0	$-3/4$	-1	$-3/4$	0	$5/4$	3

*Graphing Calculator

Mode → choose Parametric



WINDOW
Tmin=-1
Tmax=2
Tstep=0.2
Xmin=-3.5
Xmax=5
Xscl=1
Ymin=-2
Ymax=4.5
Yscl=1

the smaller the steps, the more points are plotted, smoother the curve

To find the rectangular equation when you are given the parametric equations, eliminate the parameter t through substitution.

2. Given $x(t) = 2t, y(t) = t^2 - 1$. Find the rectangular equation by eliminating the parameter.

$x = 2t$

$\frac{x}{2} = t$

$y = t^2 - 1$

$y = \left(\frac{x}{2}\right)^2 - 1$

$y = \frac{x^2}{4} - 1$

$\boxed{y = \frac{1}{4}x^2 - 1}$

3. Given the parametric equations $x(t) = 2\cos t$ and $y(t) = 2\sin t$. Eliminate the parameter.

*Recall pythagorean identity

$\sin^2 x + \cos^2 x = 1$

$x = 2\cos t$

$\frac{x}{2} = \cos t$

$y = 2\sin t$

$\frac{y}{2} = \sin t$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$\left[\frac{x^2}{4} + \frac{y^2}{4} = 1\right]4$

$\boxed{x^2 + y^2 = 4}$

Circle centered at $(0,0)$ with radius $= 2$

Derivative of a Parametric Equation

The derivative of a parametric given by $x = f(t)$ and $y = g(t)$ is found by the following:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{dy/dt}{dx/dt}, \quad dx/dt \neq 0$$

4. Given $x(t) = t^{\frac{1}{2}}$ and $y(t) = \frac{1}{4}(t^2 - 4)$ for $t \geq 0$. Find $\frac{dy}{dx}$

$y(t) = \frac{1}{4}t^2 - 1$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{\frac{1}{4}\cdot 2t - 0}{\frac{1}{2}t^{-1/2}} \rightarrow \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} \rightarrow t \cdot t^{1/2} \rightarrow \boxed{t^{3/2}}$$

as long as we know the t-value, we can calulate the slope of the curve at any moment t.

5. Given $x(t) = e^{2t}$ and $y(t) = \cos t$ for $t \geq -1$. Find the equation of a tangent line when $t = \frac{\pi}{2}$.

*Need point and slope:

$x(\pi/2) = e^{2(\pi/2)} = e^{\pi}$

$y(\pi/2) = \cos(\pi/2) = 0$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{-\sin t}{e^{2t}\cdot 2}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{-\sin(\pi/2)}{2e^{2(\pi/2)}}$$

$$\frac{dy}{dx} = \frac{-1}{2e^{\pi}}$$

$y - y_1 = m(x - x_1)$

$$\boxed{y - 0 = \frac{-1}{2e^{\pi}}(x - e^{\pi})}$$

OR

$$\boxed{y = \frac{-1}{2e^{\pi}}x + \frac{1}{2}}$$

9.1 Practice Problems:

1. For the given parametric equations, eliminate the parameter and write the corresponding rectangular equation. $x = e^{-t}$ and $y = e^{2t} - 1$.

$x = e^{-t}$

$\ln x = \ln e^{-t}$

$\ln x = -t$

$-\ln x = t$

$\ln x^{-1} = t$

$x > 0$

$y = e^{2\ln(\frac{1}{x})} - 1$

$y = e^{\ln(\frac{1}{x})^2} - 1$

$$\boxed{y = \frac{1}{x^2} - 1, \quad x > 0}$$

2. Let C be a curve described by the parametrization $x = 5t$ and $y = t^4 + 3$. Find an expression for the slope of the line tangent to C at any point (x, y).

$$\boxed{\frac{dy}{dx} = \frac{4t^3}{5}}$$

3. The position of a particle at any time $t \geq 0$ is given by $x(t) = 3t^2 + 1$ and $y(t) = \frac{2}{3}t^3$. Find $\frac{dy}{dx}$ as a function of x.

$x = 3t^2 + 1$

$x - 1 = 3t^2$

$\sqrt{\frac{x-1}{3}} = t$

$$\frac{dy}{dx} = \frac{\frac{2}{3}\cdot 3t^2}{6t}$$

$$\frac{dy}{dx} = \frac{t}{3} \leftarrow$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3}\sqrt{\frac{x-1}{3}}}$$

4. A particle moves along the curve $xy + y = 9$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dx}{dt}$?

when $x = 2$,

$2y + y = 9$

$3y = 9$

$y = 3$

$\overbrace{x}^{f}\overbrace{y}^{g} + y = 9$

$$\left(\frac{dx}{dt}\right)(y) + x\left(\frac{dy}{dt}\right) + 1\left(\frac{dy}{dt}\right) = 0$$

$$\frac{dx}{dt}(3) + 2(3) + 1(3) = 0$$

$$3\frac{dx}{dt} = -9$$

$$\boxed{\frac{dx}{dt} = -3}$$

5. A curve is described by the parametric equations $x = t\cos t$ and $y = t\sin t$. Find the equation of the line tangent to the curve at the point determined by $t = \pi$.

$x(\pi) = \pi\cos(\pi) = -\pi$

$y(\pi) = \pi\sin(\pi) = 0$

$\frac{dy}{dx} = \frac{1\sin(t) + t\cos(t)}{1\cos(t) - t\sin(t)}$

$\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin\pi + \pi\cos(\pi)}{\cos\pi - \pi\sin(\pi)}$

$= \frac{-\pi}{-1-0} = \pi$

point: $(-\pi, 0)$

slope: $m = \pi$

$\boxed{y - 0 = \pi(x + \pi)}$

6. **Calculator active.** The coordinates $(x(t), y(t))$ of the position of a drone change at rates given by $x'(t) = 2t^3$ and $y'(t) = t^{\frac{1}{2}}$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds. At what time t, for $0 \le t \le 2$, does the slope of the line tangent to its path have a slope of 1.5?

$\frac{dy}{dx} = \frac{t^{1/2}}{2t^3}$

$\frac{dy}{dx} = \frac{1}{2t^{5/2}}$

$\frac{3}{2} = \frac{1}{2t^{5/2}}$

$6t^{5/2} = 2$

$t^{5/2} = 1/3$

$t = \left(\frac{1}{3}\right)^{2/5} \approx \boxed{0.644}$

7. A curve in the xy-plane is defined by the parametric equations $x(t) = \cos(3t)$ and $y(t) = \sin(3t)$ for $t \ge 0$. What is the value of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$?

$x'(t) = -\sin(3t)\cdot 3$

$y'(t) = \cos(3t)\cdot 3$

$\sqrt{[-3\sin(3t)]^2 + [3\cos(3t)]^2}$

$\sqrt{9\sin^2(3t) + 9\cos^2(3t)} = \sqrt{9(1)} = \boxed{3}$

8. A curve is defined by the parametric equations $x(t) = at^2 + b$ and $y(t) = ct - b$, where a, b, and c are nonzero constants. What is the slope of the line tangent to the curve at the point $(x(t), y(t))$ when $t = 2$?

$x(2) = a(2)^2 + b$

$y(2) = 2c - b$

$\frac{dy}{dx} = \frac{c}{2at}$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{c}{2a(2)} = \boxed{\frac{c}{4a}}$

9. **No Calculator.** For $0 \le t \le 11$ the parametric equations $x = 3\sin t$ and $y = 2\cos t$ describe the elliptical path of an object. At the point where $t = 11$, the object travels along a line tangent to the path at that point. What is the slope of that line?

$\frac{dy}{dx} = \frac{-2\sin(t)}{3\cos(t)} = -\frac{2}{3}\tan(t)$

$\boxed{\left.\frac{dy}{dx}\right|_{t=11} = -\frac{2}{3}\tan(11)}$

10. A particle moves in the xy-plane so that its position for $t \ge 0$ is given by the parametric equations $x(t) = 2kt^2$ and $y(t) = 3t$, where k is a positive constant. When $t = 2$ the line tangent to the particle's path has a slope of 4. What is the value of k?

$\frac{dy}{dx} = \frac{3}{4kt}$

$\underbrace{\left.\frac{dy}{dx}\right|_{t=2}}_{4} = \frac{3}{4k(2)}$

$4 = \frac{3}{8k}$

$32k = 3$

$\boxed{k = 3/32}$

11. Find the equation of the line tangent to the curve defined parametrically by the equations $x(t) = t^3 + 2t$ and $y(t) = 2t^4 + 2t^2$ when $t = 1$.

$x(1) = 1^3 + 2(1) = 3$

$y(1) = 2(1)^4 + 2(1)^2 = 4$

$\frac{dy}{dx} = \frac{8t^3 + 4t}{3t^2 + 2}$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8+4}{3+2} = \frac{12}{5}$

point: $(3,4)$

slope: $m = 12/5$

$\boxed{y - 4 = \frac{12}{5}(x-3)}$

12. For what values of t does the curve given by the parametric equations $x(t) = \frac{1}{4}t^4 - \frac{9}{2}t^2$ and $y(t) = 3t^3 + 2t$ have a vertical tangent? Set $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 4 \cdot \frac{1}{4}t^3 - \frac{9}{2} \cdot 2t$

$0 = t^3 - 9t$

$0 = t(t^2 - 9)$

$t = 0, \pm 3$

$\boxed{t = 0, -3, 3}$

13. Suppose a curve is given by the parametric equations $x = f(t)$ and $y = g(t)$, for all $t > 1$ and $\frac{dy}{dt} = \frac{t^2+2}{t-1} * \frac{dx}{dt}$. What is the value of $\frac{dy}{dx}$ when $t = 2$?

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \rightarrow \frac{\frac{t^2+2}{t-1} \cdot \cancel{\frac{dx}{dt}}}{\cancel{dx/dt}}$

$\frac{dy}{dx} = \frac{t^2+2}{t-1}$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{2^2+2}{2-1} = \frac{6}{1} = \boxed{6}$

9.1 Parametric Equations

Test Prep

14. A curve is defined parametrically by $x(t) = t^2$ and $y(t) = t^3 - 3t$. Find the points on the graph where the tangent line is horizontal or vertical.

$\frac{dx}{dt} = 2t$

$0 = 2t$

$t = 0$

(vertical tangent)

$x(0) = 0$

$y(0) = 0$

$\rightarrow$ at $\boxed{(0,0)}$

$\frac{dy}{dt} = 3t^2 - 3$

$0 = 3(t^2 - 1)$ ← to find horizontal tangent

$t = 1, t = -1$

$x(1) = 1^2 = 1$

$y(1) = 1^3 - 3(1) = -2$

$x(-1) = 1^2 = 1$

$y(-1) = (-1)^3 - 3(-1) = 2$

$\boxed{\text{at } (1,-2) \text{ and } (1,2)}$

15. **Free Response**. Consider the curve given by the parametric equations $y = t^3 - 12t$ and $x = \frac{1}{2}t^2 - t$.

a. Find $\frac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{3t^2-12}{2\cdot\frac{1}{2}t-1} = \boxed{\frac{3t^2-12}{t-1}}$$

b. Write an equation for the line tangent to the curve at the point where $t = -1$.

$y(-1) = (-1)^3 - 12(-1) = 11$

$x(-1) = \frac{1}{2}(-1)^2 - (-1) = 3/2$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2-12}{-1-1} = \frac{-9}{-2} = \frac{9}{2}$$

point: $(3/2, 11)$

slope: $m = 9/2$

$$\boxed{y - 11 = \frac{9}{2}(x - 3/2)}$$

c. Find the x and y coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

$\frac{dx}{dt} = t - 1$

$0 = t - 1$

$t = 1$

(vertical tangent)

$x(1) = \frac{1}{2} - 1 = -1/2$

$y(1) = 1 - 12 = -11$

$\frac{dy}{dt} = 3t^2 - 12$ (horizontal tangent)

$0 = 3(t^2-4)$

$t = 2, t = -2$

$x(2) = \frac{1}{2}(2)^2 - 2 = 0$

$y(2) = (2)^3 - 12(2) = -16$

$x(-2) = \frac{1}{2}(-2)^2 - (-2) = 4$

$y(-2) = (-2)^3 - 12(-2) = 16$

Horizontal tangent at $(0, -16)$ and $(4, 16)$

Vertical tangent at $(-1/2, -11)$

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16. A curve is given by the parametric equations $x(t) = 5t^3 - 5$ and $y(t) = t^2 + 7$. What is the equation of the tangent line to the curve when $t = 1$?

$x(1) = 5(1)^3 - 5 = 0$

$y(1) = 1^2 + 7 = 8$

$$\frac{dy}{dx} = \frac{2t}{15t^2}$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{15}$$

point: $(0, 8)$

slope: $m = 2/15$

$$\boxed{y - 8 = \frac{2}{15}(x - 0)}$$

A. $x = 0$ B. $\boxed{y = \frac{2}{15}x + 8}$ C. $y = \frac{2}{15}x + 1$ D. $y = 8$ E. $y = \frac{15}{2}x + 7$