Calculus BC Ch. 9.1 Notes Sequences

-List the terms of a sequence and write a sequence.

-Determine whether a sequence converges or diverges

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers (**natural numbers**). Although a sequence is a function, it is common to represent sequences by subscript notation.

Example: 1, 2, 3, 4, 5, ... n, ...

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & \ldots & n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & \ldots & a_n & \ldots \end{array}$$

is a representation of a sequence.

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number a_n is called the **nth term** of the sequence, and the entire sequence is called $\{a_n\}$.

The sequence can be expressed by either:

1) A number of terms in the sequence, separated by commas
2) An explicit function defined by the rule of sequence
3) Rule of sequence set off in braces

Example 1:

If $a_n = \left\{\dfrac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim\limits_{n\to\infty} a_n$

Convergent sequence: list of numbers in a specified order that has a limit (approaching a real number)

Divergent sequence: list of numbers in a specified order that do not have a limit

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim\limits_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim\limits_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) If $\lim\limits_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim\limits_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

$n!$ is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

The primary focus of this chapter concerns sequences whose terms approach limiting values. These sequences are said to **converge**.

Example 2:

Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

(c) $a_n = 3 + (-1)^n$

(d) $a_n = \frac{n}{1-2n}$

(e) $a_n = \frac{\ln n}{n}$

(f) $a_n = \frac{n!}{(n+2)!}$

(g) $a_n = \frac{2n!}{(n-1)!}$

(h) $a_n = \frac{n+(-1)^n}{n}$

(i) $a_n = \frac{(-1)^n (n-1)}{n}$

(j) $a_n = \frac{2^n}{(n+1)!}$

(k) $a_n = \left(1+\frac{1}{n}\right)^n$

(l) $\left\{\frac{(2n)!}{n^n}\right\}$

Example 3: Find the nth Term of a Sequence

(a) 3, 8, 13, 18, . . .

(b) 5, -15, 45, -135, . . .

(c) 1, 4, 9, 16, 25, . . .

(d) 4, 10, 28, 82, . . .

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

(f) $\ln 1, \ln 2, \ln 4, \ln 8, \ldots$

g) Find a sequence $\{a_n\}$ whose first five terms are $\frac{-2}{1}, \frac{8}{2}, \frac{-26}{6}, \frac{80}{24}, \frac{-242}{120}$ ... then determine if it converges.

Key

Calculus BC 9.1 Notes Sequences

-List the terms of a sequence and write a sequence.

-Determine whether a sequence converges or diverges

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers (**natural numbers**) . Although a sequence is a function, it is common to represent sequences by subscript notation.

Example: 1, 2, 3, 4, 5, ... n, ...

$$\begin{matrix} 1, & 2, & 3, & 4, & 5, & \dots & n, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_n & \dots \end{matrix}$$

is a representation of a sequence.

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers $a_1, a_2, a_3, \dots, a_n, \dots$ are the **terms** of the sequence. The number a_n is called the **nth term** of the sequence, and the entire sequence is called $\{a_n\}$.

The sequence can be expressed by either:

1) A number of terms in the sequence, separated by commas
2) An explicit function defined by the rule of sequence
3) Rule of sequence set off in braces

Example 1:

Rule of sequence

If $a_n = \left\{\frac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim\limits_{n\to\infty} a_n$

$a_n = \frac{4}{5}, \frac{8}{7}, \frac{12}{9}, \frac{16}{11}, \frac{20}{13}, \dots$

*what happens to the terms as we go out to infinity?

$\lim\limits_{n\to\infty} \frac{4n}{3+2n} = \frac{4}{2} = \boxed{2}$ ← approaches the ratio of the leading coefficients.

Notes:

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim\limits_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim\limits_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) * If $\lim\limits_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim\limits_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

$n!$ is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$

The primary focus of this chapter concerns sequences whose terms approach limiting values. These sequences are said to **converge**.

convergent sequence: has a limit (approaches a real number)

Example 2: divergent sequence: does not have a limit.

Determine whether the following sequences converge or diverge.

$a_n = 2, 4, 2, 4, 2, 4, \ldots$

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$ so sequence converges to 1

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$, therefore sequence converges to 0.

(c) $a_n = 3 + (-1)^n$

Sequence diverges since $\lim_{n\to\infty} a_n$ oscillates between 2 and 4.

(d) $a_n = \frac{n}{1-2n}$

converges to $\frac{-1}{2}$

$\lim_{n\to\infty} \frac{n}{1-2n} = -\frac{1}{2}$

(e) $a_n = \frac{\ln n}{n}$

converges to 0 since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

*consider comparative growth rate. $L < R < P < E$

(f) $a_n = \frac{n!}{(n+2)!} = \frac{n!}{(n+2)(n+1)n!}$

$\lim_{n\to\infty} \frac{1}{(n+2)(n+1)} = 0$

sequence converges to 0.

g) $\{a_n\} = \frac{2n!}{(n-1)!} =$

$\lim_{n\to\infty} \frac{2n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = \infty$

sequence diverges to $+\infty$

h) $\{a_n\} = \frac{n + (-1)^n}{n}$

$a_n = 1 + \frac{(-1)^n}{n} \to \lim_{n\to\infty}\left[1 + \frac{(-1)^n}{n}\right]$

$= 1 + \lim_{n\to\infty} \frac{(-1)^n}{n} = 0 = 1$ sequence converges to 1.

i) $\{a_n\} = \frac{(-1)^n(n-1)}{n}$

$\lim_{n\to\infty} \frac{n-1}{n} = \pm 1$

$0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}$

diverges by oscillation (approaching 1 and -1)

(j) $a_n = \frac{2^n}{(n+1)!}$

*use calculator List→ops→seq.

$seq(2^x/(x+1)!, x, 1, 99)$

$\lim_{n\to\infty} \frac{2^n}{(n+1)!} = 0$

sequence converges to zero.

(k) $a_n = \left(1 + \frac{1}{n}\right)^n$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \boxed{e}$

converges to

(l) $\left\{\frac{(2n)!}{n^n}\right\}$

$\lim_{n\to\infty} a_n = \infty$

sequence diverges.

$\log x < \sqrt{x} < x^n < b^x < x^x < n!$

Example 3: Find the nth Term of a Sequence

(a) 3, 8, 13, 18, . . . $d = 5$

*arithmetic sequence

$a_n = a_1 + (n-1)d$

$a_n = 3 + (n-1)d$

$a_n = 3 + (n-1)(5)$

$\boxed{a_n = 5n - 2}$

diverges

(b) 5, -15, 45, -135, . . . $r = -3$

*geometric sequence

$a_n = a_1 \cdot r^{n-1}$

$a_n = 5(-3)^{n-1}$, diverges

(c) 1, 4, 9, 16, 25, . . .

$a_n = n^2$, diverges

(d) 4, 10, 28, 82, . . .

$a_n = 3^n + 1$

diverges

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

$a_n = \frac{n+1}{2n-1}$

$\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2}$. converges to $\frac{1}{2}$.

(f) ln1, ln2, ln4, ln8,. . .

$a_n = \ln(2^{n-1})$

diverges to ∞.

g) Find a sequence $\{a_n\}$ whose first five terms are $\frac{-2}{1}, \frac{8}{2}, \frac{-26}{6}, \frac{80}{24}, \frac{-242}{120}$ ... then determine if it converges.

$a_n = (-1)^n\left(\frac{3^n - 1}{n!}\right)$

$\lim_{n\to\infty} \frac{3^n - 1}{n!} = 0$

sequence converges to 0.