CHAPTER 9
Infinite Series

Section 9.1 Sequences

1. $a_n = 3^n$

$a_1 = 3^1 = 3$

$a_2 = 3^2 = 9$

$a_3 = 3^3 = 27$

$a_4 = 3^4 = 81$

$a_5 = 3^5 = 243$

2. $a_n = \left(-\frac{2}{5}\right)^n$

$a_1 = \left(-\frac{2}{5}\right)^1 = -\frac{2}{5}$

$a_2 = \left(-\frac{2}{5}\right)^2 = \frac{4}{25}$

$a_3 = \left(-\frac{2}{5}\right)^3 = -\frac{8}{125}$

$a_4 = \left(-\frac{2}{5}\right)^4 = \frac{16}{625}$

$a_5 = \left(-\frac{2}{5}\right)^5 = -\frac{32}{3125}$

3. $a_n = \sin \frac{n\pi}{2}$

$a_1 = \sin \frac{\pi}{2} = 1$

$a_2 = \sin \pi = 0$

$a_3 = \sin \frac{3\pi}{2} = -1$

$a_4 = \sin 2\pi = 0$

$a_5 = \sin \frac{5\pi}{2} = 1$

4. $a_n = \frac{3n}{n+4}$

$a_1 = \frac{3(1)}{1+4} = \frac{3}{5}$

$a_2 = \frac{3(2)}{2+4} = \frac{6}{6} = 1$

$a_3 = \frac{3(3)}{3+4} = \frac{9}{7}$

$a_4 = \frac{3(4)}{4+4} = \frac{12}{8} = \frac{3}{2}$

$a_5 = \frac{3(5)}{5+4} = \frac{15}{9} = \frac{5}{3}$

5. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

$a_1 = \frac{2}{1} = 2$

$a_2 = -\frac{2}{2} = -1$

$a_3 = \frac{2}{3}$

$a_4 = -\frac{2}{4} = -\frac{1}{2}$

$a_5 = \frac{2}{5}$

6. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

$a_1 = 2 + 2 - 1 = 3$

$a_2 = 2 + 1 - \frac{1}{4} = \frac{11}{4}$

$a_3 = 2 + \frac{2}{3} - \frac{1}{9} = \frac{23}{9}$

$a_4 = 2 + \frac{2}{4} - \frac{1}{16} = \frac{39}{16}$

$a_5 = 2 + \frac{2}{5} - \frac{1}{25} = \frac{59}{25}$

7. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$a_2 = 2(a_1 - 1)$

$= 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1)$

$= 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1)$

$= 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1)$

$= 2(10 - 1) = 18$

8. $a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$

$a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$

$a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$

$a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$

$a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768)^2 = 196,608$

9. $a_n = \dfrac{10}{n+1}, a_1 = \dfrac{10}{1+1} = 5, a_2 = \dfrac{10}{3}$

Matches (c).

10. $a_n = \dfrac{10n}{n+1}, a_1 = \dfrac{10}{2} = 5, a_2 = \dfrac{20}{3}$

Matches (a).

11. $a_n = (-1)^n, a_1 = -1, a_2 = 1, a_3 = -1, \ldots$

Matches (d).

12. $a_n = \dfrac{(-1)^n}{n}, a_1 = \dfrac{-1}{1} = -1, a_2 = \dfrac{1}{2}.$

Matches (b).

13. $a_n = 3n - 1$

$a_5 = 3(5) - 1 = 14$

$a_6 = 3(6) - 1 = 17$

Add 3 to preceding term.

14. $a_n = 3 + 5n$

$a_6 = 3 + 5(6) = 33$

$a_7 = 3 + 5(7) = 38$

Add 5 to preceding term.

15. $a_{n+1} = 2a_n, a_1 = 5$

$a_5 = 2(40) = 80$

$a_6 = 2(80) = 160$

Multiply the preceding term by 2.

16. $a_n = -\frac{1}{3}a_{n-1}, a_1 = 6$

$a_5 = -\frac{1}{3}\left(-\frac{2}{9}\right) = \frac{2}{27}$

$a_6 = -\frac{1}{3}\left(\frac{2}{27}\right) = -\frac{2}{81}$

Multiply the preceding term by $-\frac{1}{3}$.

17. $\dfrac{(n+1)!}{n!} = \dfrac{n!(n+1)}{n!} = n + 1$

18. $\dfrac{n!}{(n+2)!} = \dfrac{n!}{(n+2)(n+1)n!} = \dfrac{1}{(n+2)(n+1)}$

19. $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n-1)!(2n)(2n+1)} = \dfrac{1}{2n(2n+1)}$

20. $\dfrac{(2n+2)!}{(2n)!} = \dfrac{(2n)!(2n+1)(2n+2)}{(2n)!}$

$= (2n+1)(2n+2)$

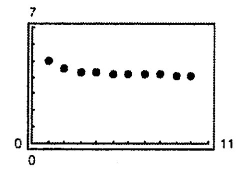
21. $\lim_{n\to\infty} \dfrac{5n^2}{n^2+2} = 5$

22. $\lim_{n\to\infty}\left(6 + \dfrac{2}{n^2}\right) = 6 + 0 = 6$

23. $\lim_{n\to\infty} \dfrac{2n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \dfrac{2}{\sqrt{1 + \left(1/n^2\right)}} = \dfrac{2}{1} = 2$

24. $\lim_{n\to\infty} \cos\left(\dfrac{2}{n}\right) = 1$

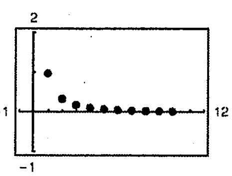
25.



The graph seems to indicate that the sequence converges to 4. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n+1}{n} = \lim_{x\to\infty} \frac{4x+1}{x} = 4.$$

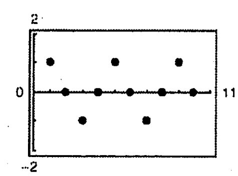
26.



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}} = 0.$$

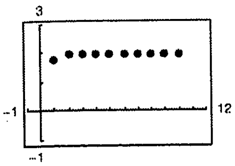
27.



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$\{a_n\} = \{1, 0, -1, 0, 1, \ldots\}.$

So, $\lim_{n\to\infty} a_n$ does not exist.

28. 

The graph seems to indicate that the sequence converges to 2. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 - \frac{1}{4^n}\right) = 2 - 0 = 2.$$

29. $\lim_{n\to\infty} \frac{5}{n+2} = 0$, converges

30. $\lim_{n\to\infty}\left(8 + \frac{5}{n}\right) = 8 + 0 = 8$, converges

31. $\lim_{n\to\infty} (-1)^n\left(\frac{n}{n+1}\right)$

does not exist (oscillates between −1 and 1), diverges.

32. $\lim_{n\to\infty} \frac{1 + (-1)^n}{n^2} = 0$, converges

33. $\lim_{n\to\infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n\to\infty} \frac{10 + 3/n + 7/n^2}{2 - 6/n^2}$

$= \frac{10}{2} = 5$, converges

34. $\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$, converges

35. $\lim_{n\to\infty} \frac{\ln(n^3)}{2n} = \lim_{n\to\infty} \frac{3}{2}\frac{\ln(n)}{n}$

$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{1}{n}\right) = 0$, converges

(L'Hôpital's Rule)

36. $\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty}\left(\frac{5}{3}\right)^n = \infty$, diverges

37. $\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$, diverges

38. $\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)} = 0$, converges

39. $\lim_{n\to\infty} \frac{n^p}{e^n} = 0$, converges

$(p > 0, n \ge 2)$

40. $a_n = n \sin \frac{1}{n}$

Let $f(x) = x \sin \frac{1}{x}$.

$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}$

$= \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2}$

$= \lim_{x\to\infty} \cos \frac{1}{x} = \cos 0$

= 1 (L'Hôpital's Rule)

or,

$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$. Therefore,

$\lim_{n\to\infty} n \sin \frac{1}{n} = 1$, converges.

41. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, converges

42. $\lim_{n\to\infty} -3^{-n} = \lim_{n\to\infty} \frac{-1}{3^n} = 0$, converges

43. $\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} (\sin n)\frac{1}{n} = 0$,

converges (because $(\sin n)$ is bounded)

44. $\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = 0$, converges

45. $a_n = -4 + 6n$

46. $a_n = \frac{1}{n!}$

47. $a_n = n^2 - 3$

48. $a_n = \frac{(-1)^{n-1}}{n^2}$

49. $a_n = \frac{n+1}{n+2}$

50. $a_n = (2n)!, n = 1, 2, 3, \ldots$

51. $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

52. $a_n = \frac{n}{(n+1)(n+2)}$

53. $a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1},$

Monotonic; $|a_n| < 4$, bounded

54. Let $f(x) = \frac{3x}{x+2}$. Then $f'(x) = \frac{6}{(x+2)^2}$.

So, f is increasing which implies $\{a_n\}$ is increasing.

$|a_n| < 3$, bounded

55. $a_n = ne^{-n/2}$

$a_1 = 0.6065$

$a_2 = 0.7358$

$a_3 = 0.6694$

Not monotonic; $|a_n| \le 0.7358$, bounded

56. $a_n = \left(-\frac{2}{3}\right)^n$

$a_1 = -\frac{2}{3}$

$a_2 = \frac{4}{9}$

$a_3 = -\frac{8}{27}$

Not monotonic; $|a_n| \le \frac{2}{3}$, bounded

57. $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic; $|a_n| \le \frac{2}{3}$, bounded

58. $a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$

Monotonic; $\lim_{n\to\infty} a_n = \infty$, not bounded

59. $a_n = \sin\left(\frac{n\pi}{6}\right)$

$a_1 = 0.500$

$a_2 = 0.8660$

$a_3 = 1.000$

$a_4 = 0.8660$

Not monotonic; $|a_n| \le 1$, bounded

60. $a_n = \frac{\cos n}{n}$

$a_1 = 0.5403$

$a_2 = -0.2081$

$a_3 = -0.3230$

$a_4 = -0.1634$

Not monotonic; $|a_n| \le 1$, bounded

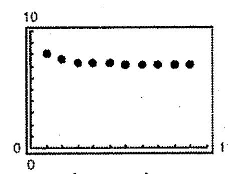
61. (a) $a_n = 7 + \frac{1}{n}$

$\left|7 + \frac{1}{n}\right| \le 8 \Rightarrow \{a_n\}$, bounded

$a_n = 7 + \frac{1}{n} > 7 + \frac{1}{n+1} = a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)



$\lim_{n\to\infty}\left(7 + \frac{1}{n}\right) = 7$

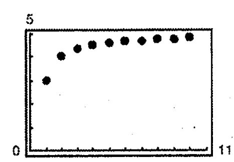
62. (a) $a_n = 5 - \frac{2}{n}$

$\left|5 - \frac{2}{n}\right| \le 5 \Rightarrow \{a_n\}$, bounded

$a_n = 5 - \frac{2}{n} < 5 - \frac{2}{n+1} = a_n + 1 \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)



$\lim_{n\to\infty}\left(5 - \frac{2}{n}\right) = 5 - 0 = 5$

63. (a) $a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right)$

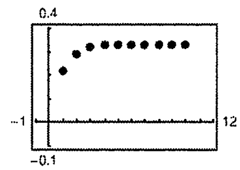
$\left|\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right| < \frac{1}{3} \Rightarrow \{a_n\}$, bounded

$a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right) < \frac{1}{3}\left(1 - \frac{1}{3^{n+1}}\right)$

$= a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left[\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right] = \frac{1}{3}$$

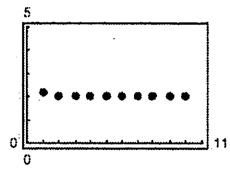
64. (a) $a_n = 2 + \frac{1}{5^n}$

$\left|2 + \frac{1}{5^n}\right| < 3 \Rightarrow \{a_n\}$, bounded

$a_n = 2 + \frac{1}{5^n} > 2 + \frac{1}{5^{n+1}} = a_{n+1} \Rightarrow \{a_n\}$, monotonic

Therefore, $\{a_n\}$ converges.

(b)



$$\lim_{n\to\infty}\left(2 + \frac{1}{5^n}\right) = 2 + 0 = 2$$

65. $\{a_n\}$ has a limit because it is a bounded, monotonic sequence. The limit is less than or equal to 4, and greater than or equal to 2.

$2 \le \lim_{n\to\infty} a_n \le 4$

66. The sequence $\{a_n\}$ could converge or diverge. If $\{a_n\}$ is increasing, then it converges to a limit less than or equal to 1. If $\{a_n\}$ is decreasing, then it could converge (example: $a_n = 1/n$) or diverge (example: $a_n = -n$).

67. $A_n = P\left(1 + \frac{r}{12}\right)^n$

(a) Because $P > 0$ and $\left(1 + \frac{r}{12}\right) > 1$, the sequence diverges. $\lim_{n\to\infty} A_n = \infty$

(b) $P = 10{,}000$, $r = 0.055$, $A_n = 10{,}000\left(1 + \frac{0.055}{12}\right)^n$.

$A_0 = 10{,}000$
$A_1 = 10{,}045.83$
$A_2 = 10{,}091.88$
$A_3 = 10{,}138.13$
$A_4 = 10{,}184.60$
$A_5 = 10{,}231.28$
$A_6 = 10{,}278.17$
$A_7 = 10{,}325.28$
$A_8 = 10{,}372.60$
$A_9 = 10{,}420.14$
$A_{10} = 10{,}467.90$

68. (a) $A_n = 100(401)(1.0025^n - 1)$

$A_0 = 0$
$A_1 = 100.25$
$A_2 = 200.75$
$A_3 = 301.50$
$A_4 = 402.51$
$A_5 = 503.76$
$A_6 = 605.27$

(b) $A_{60} = 6480.83$

(c) $A_{240} = 32,912.28$

69. No, it is not possible. See the "Definition of the Limit of a sequence". The number L is unique.

70. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit. See the definition.

(c) A sequence is monotonic if its terms are nondecreasing, or nonincreasing.

(d) A sequence is bounded if it is bounded below $(a_n \geq N$ for some $N)$ and bounded above $(a_n \leq M$ for some $M)$.

71. (a) $a_n = 10 - \frac{1}{n}$

(b) Impossible. The sequence converges by Theorem 9.5.

(c) $a_n = \frac{3n}{4n+1}$

(d) Impossible. An unbounded sequence diverges.

72. The graph on the left represents a sequence with alternating signs because the terms alternate from being above the x-axis to being below the x-axis.

73. (a) $A_n = (0.8)^n\, 4{,}500{,}000{,}000$

(b) $A_1 = \$3,600,000,000$
$A_2 = \$2,880,000,000$
$A_3 = \$2,304,000,000$
$A_4 = \$1,843,200,000$

(c) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} (0.8)^n(4.5) = 0$, converges

74. $P_n = 25{,}000(1.045)^n$

$P_1 = \$26,125.00$
$P_2 = \$27,300.63$
$P_3 = \$28,529.15$
$P_4 = \$29,812.97$
$P_5 = \$31,154.55$

75. $a_n = \sqrt[n]{n} = n^{1/n}$

$a_1 = 1^{1/1} = 1$
$a_2 = \sqrt{2} \approx 1.4142$
$a_3 = \sqrt[3]{3} \approx 1.4422$
$a_4 = \sqrt[4]{4} \approx 1.4142$
$a_5 = \sqrt[5]{5} \approx 1.3797$
$a_6 = \sqrt[6]{6} \approx 1.3480$

Let $y = \lim_{n\to\infty} n^{1/n}$.

$$\ln y = \lim_{n\to\infty}\left(\frac{1}{n}\ln n\right) = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{n} = 0$$

Because $\ln y = 0$, you have $y = e^0 = 1$. Therefore, $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

76. $a_n = \left(1 + \frac{1}{n}\right)^n$

$a_1 = 2.0000$
$a_2 = 2.2500$
$a_3 \approx 2.3704$
$a_4 \approx 2.4414$
$a_5 \approx 2.4883$
$a_6 \approx 2.5216$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

77. Because

$\lim_{n\to\infty} s_n = L > 0,$

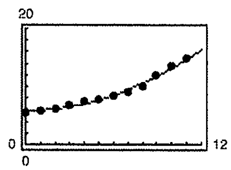
there exists for each $\varepsilon > 0$,

an integer N such that $|s_n - L| < \varepsilon$ for every $n > N$.

Let $\varepsilon = L > 0$ and you have,

$|s_n - L| < L, -L < s_n - L < L$, or $0 < s_n < 2L$ for each $n > N$.

78. (a) $a_n = 0.072n^2 + 0.02n + 5.8$



(b) For 2020, $n = 20$: $a_{20} = \$35$ trillion

79. True

80. True

81. True

82. False. Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ then $\{a_n\}$ and $\{b_n\}$ diverge. But $\{a_n + b_n\} = \{(-1)^n + (-1)^{n+1}\}$ converges to 0.

83. $a_{n+2} = a_n + a_{n+1}$

(a)
$a_1 = 1$ $\quad a_7 = 8 + 5 = 13$
$a_2 = 1$ $\quad a_8 = 13 + 8 = 21$
$a_3 = 1 + 1 = 2$ $\quad a_9 = 21 + 13 = 34$
$a_4 = 2 + 1 = 3$ $\quad a_{10} = 34 + 21 = 55$
$a_5 = 3 + 2 = 5$ $\quad a_{11} = 55 + 34 = 89$
$a_6 = 5 + 3 = 8$ $\quad a_{12} = 89 + 55 = 144$

(b) $b_n = \dfrac{a_{n+1}}{a_n}, n \geq 1$

$b_1 = \dfrac{1}{1} = 1$ $\qquad b_6 = \dfrac{13}{8} = 1.625$

$b_2 = \dfrac{2}{1} = 2$ $\qquad b_7 = \dfrac{21}{13} \approx 1.6154$

$b_3 = \dfrac{3}{2} = 1.5$ $\qquad b_8 = \dfrac{34}{21} \approx 1.6190$

$b_4 = \dfrac{5}{3} \approx 1.6667$ $\qquad b_9 = \dfrac{55}{34} \approx 1.6176$

$b_5 = \dfrac{8}{5} = 1.6$ $\qquad b_{10} = \dfrac{89}{55} \approx 1.6182$

(c) $$1 + \frac{1}{b_{n-1}} = 1 + \frac{1}{a_n/a_{n-1}} = 1 + \frac{a_{n-1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n$$

(d) If $\lim_{n\to\infty} b_n = \rho$, then $\lim_{n\to\infty}\left(1 + \dfrac{1}{b_{n-1}}\right) = \rho$.

Because $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n-1}$, you have

$1 + (1/\rho) = \rho$.

$\rho + 1 = \rho^2$

$0 = \rho^2 - \rho - 1$

$\rho = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Because a_n, and therefore b_n, is positive,

$\rho = \dfrac{1 + \sqrt{5}}{2} \approx 1.6180.$

84. Let $f(x) = \sin(\pi x)$

$\lim_{x\to\infty} \sin(\pi x)$ does not exist.

$a_n = f(n) = \sin(\pi n) = 0$ for all n

$\lim_{n\to\infty} a_n = 0$, coverges

85. (a) $a_1 = \sqrt{2} \approx 1.4142$

$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$

$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$

$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$

$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$

(b) $a_n = \sqrt{2 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{2}$

(c) First use mathematical induction to show that $a_n \le 2$; clearly $a_1 \le 2$. So assume $a_k \le 2$. Then

$$a_k + 2 \le 4$$
$$\sqrt{a_k + 2} \le 2$$
$$a_{k+1} \le 2.$$

Now show that $\{a_n\}$ is an increasing sequence. Because $a_n \ge 0$ and $a_n \le 2$,

$$(a_n - 2)(a_n + 1) \le 0$$
$$a_n^2 - a_n - 2 \le 0$$
$$a_n^2 \le a_n + 2$$
$$a_n \le \sqrt{a_n + 2}$$
$$a_n \le a_{n+1}.$$

Because $\{a_n\}$ is a bounding increasing sequence, it converges to some number L, by Theorem 9.5.

$$\lim_{n\to\infty} a_n = L \Rightarrow \sqrt{2 + L} = L \Rightarrow 2 + L = L^2 \Rightarrow L^2 - L - 2 = 0$$
$$\Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = 2 \quad (L \ne -1)$$

86. (a) Use mathematical induction to show that

$$a_n \le \frac{1 + \sqrt{1 + 4k}}{2}.$$

[Note that if $k = 2$, and $a_n \le 3$, and if $k = 6$, then $a_n \le 3$.] Clearly,

$$a_1 = \sqrt{k} \le \frac{\sqrt{1 + 4k}}{2} \le \frac{1 + \sqrt{1 + 4k}}{2}.$$

Before proceeding to the induction step, note that

$$2 + 2\sqrt{1 + 4k} + 4k = 2 + 2\sqrt{1 + 4k} + 4k$$
$$\frac{1 + \sqrt{1 + 4k}}{2} + k = \frac{1 + 2\sqrt{1 + 4k} + 1 + 4k}{4}$$
$$\frac{1 + \sqrt{1 + 4k}}{2} + k = \left[\frac{1 + \sqrt{1 + 4k}}{2}\right]^2$$
$$\sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} = \frac{1 + \sqrt{1 + 4k}}{2}.$$

So assume $a_n \le \dfrac{1 + \sqrt{1 + 4k}}{2}$. Then

$$a_n + k \le \frac{1 + \sqrt{1 + 4k}}{2} + k$$
$$\sqrt{a_n + k} \le \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k}$$
$$a_{n+1} \le \frac{1 + \sqrt{1 + 4k}}{2}.$$

$\{a_n\}$ is increasing because

$$\left(a_n - \frac{1 + \sqrt{1 + 4k}}{2}\right)\left(a_n - \frac{1 - \sqrt{1 + 4k}}{2}\right) \le 0$$
$$a_n^2 - a_n - k \le 0$$
$$a_n^2 \le a_n + k$$
$$a_n \le \sqrt{a_n + k}$$
$$a_n \le a_{n+1}.$$

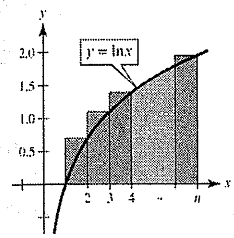
(b) Because $\{a_n\}$ is bounded and increasing, it has a limit L.

(c) $\lim_{n\to\infty} a_n = L$ implies that

$$L = \sqrt{k + L} \Rightarrow L^2 = k + L$$
$$\Rightarrow L^2 - L - k = 0$$
$$\Rightarrow L = \frac{1 \pm \sqrt{1 + 4k}}{2}.$$

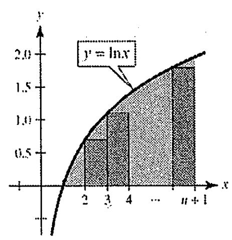
Because $L > 0$, $L = \dfrac{1 + \sqrt{1 + 4k}}{2}$.

87. (a)



$$\int_1^n \ln x\,dx < \ln 2 + \ln 3 + \cdots + \ln n$$
$$= \ln(1 \cdot 2 \cdot 3 \cdots n) = \ln(n!)$$

(b)



$$\int_1^{n+1} \ln x\,dx > \ln 2 + \ln 3 + \cdots + \ln n = \ln(n!)$$

(c) $\int \ln x\,dx = x \ln x - x + C$

$$\int_1^n \ln x\,dx = n \ln n - n + 1 = \ln n^n - n + 1$$

From part (a): $\ln n^n - n + 1 < \ln(n!)$

$$e^{\ln n^n - n + 1} < n!$$
$$\frac{n^n}{e^{n-1}} < n!$$

$$\int_1^{n+1} \ln x\,dx = (n + 1)\ln(n + 1) - (n + 1) + 1$$
$$= \ln(n + 1)^{n+1} - n$$

From part (b): $\ln(n + 1)^{n+1} - n > \ln(n!)$

$$e^{\ln(n+1)^{n+1} - n} > n!$$
$$\frac{(n + 1)^{n+1}}{e^n} > n!$$

(d) $$\frac{n^n}{e^{n-1}} < n! < \frac{(n + 1)^{n+1}}{e^n}$$

$$\frac{n}{e^{1-(1/n)}} < \sqrt[n]{n!} < \frac{(n + 1)^{(n+1)/n}}{e}$$

$$\frac{1}{e^{1-(1/n)}} < \frac{\sqrt[n]{n!}}{n} < \frac{(n + 1)^{1+(1/n)}}{ne}$$

$$\lim_{n\to\infty} \frac{1}{e^{1-(1/n)}} = \frac{1}{e}$$

$$\lim_{n\to\infty} \frac{(n + 1)^{1+(1/n)}}{ne} = \lim_{n\to\infty} \frac{(n + 1)}{n}\frac{(n + 1)^{1/n}}{e}$$
$$= (1)\frac{1}{e}$$
$$= \frac{1}{e}$$

By the Squeeze Theorem, $\lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

(e) $n = 20$: $\dfrac{\sqrt[20]{20!}}{20} \approx 0.4152$

$n = 50$: $\dfrac{\sqrt[50]{50!}}{50} \approx 0.3897$

$n = 100$: $\dfrac{\sqrt[100]{100!}}{100} \approx 0.3799$

$\dfrac{1}{e} \approx 0.3679$

88. For a given $\varepsilon > 0$, you must find $M > 0$ such that

$$|a_n - L| = \left|\frac{1}{n^3}\right| < \varepsilon$$

whenever $n > M$. That is,

$$n^3 > \frac{1}{\varepsilon} \text{ or } n > \left(\frac{1}{\varepsilon}\right)^{1/3}.$$

So, let $\varepsilon > 0$ be given. Let M be an integer satisfying $M > (1/\varepsilon)^{1/3}$. For $n > M$, you have

$$n > \left(\frac{1}{\varepsilon}\right)^{1/3}$$

$$n^3 > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{n^3} \Rightarrow \left|\frac{1}{n^3} - 0\right| < \varepsilon.$$

So, $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

89. For a given $\varepsilon > 0$, you must find $M > 0$ such that $|a_n - L| = |r^n| \varepsilon$ whenever $n > M$. That is,

$n \ln|r| < \ln(\varepsilon)$ or

$$n > \frac{\ln(\varepsilon)}{\ln|r|} \text{ (because } \ln|r| < 0 \text{ for } |r| < 1\text{)}.$$

So, let $\varepsilon > 0$ be given. Let M be an integer satisfying

$$M > \frac{\ln(\varepsilon)}{\ln|r|}.$$

For $n > M$, you have

$$n > \frac{\ln(\varepsilon)}{\ln|r|}$$

$$n \ln|r| < \ln(\varepsilon)$$

$$\ln|r|^n < \ln(\varepsilon)$$

$$|r|^n < \varepsilon$$

$$|r^n - 0| < \varepsilon.$$

So, $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

90. Answers will vary. *Sample answer*:

$\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$ diverges

$\{a_{2n}\} = \{(-1)^{2n}\} = \{1, 1, 1, 1, \ldots\}$ converges

91. If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$. Then $-a_1 \le -a_2 \le -a_3 \le \cdots \le -a_n \le \cdots$ is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Because $\{-a_n\}$ converges, then so does $\{a_n\}$.

92. Define $a_n = \dfrac{x_{n+1} + x_{n-1}}{x_n}$, $n \ge 1$.

$$x_{n+1}^2 - x_n x_{n+2} = 1 = x_n^2 - x_{n-1}x_{n+1} \Rightarrow$$

$$x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_n + x_{n+2})$$

$$\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_{n+2} + x_n}{x_{n+1}}$$

$$a_n = a_{n+1}$$

Therefore, $a_1 = a_2 = \ldots = a$. So,

$$x_{n+1} = a_n x_n - x_{n-1} = a x_n - x_{n-1}.$$