9.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Listing the Terms of a Sequence In Exercises 1–6, write the first five terms of the sequence.

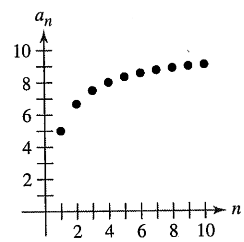
1. $a_n = 3^n$
2. $a_n = \left(-\frac{2}{5}\right)^n$
3. $a_n = \sin \frac{n\pi}{2}$
4. $a_n = \frac{3n}{n+4}$
5. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$
6. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

Listing the Terms of a Sequence In Exercises 7 and 8, write the first five terms of the recursively defined sequence.

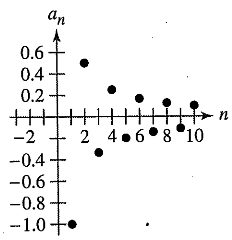
7. $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$
8. $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k^2$

Matching In Exercises 9–12, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



(b)



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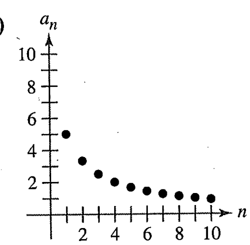

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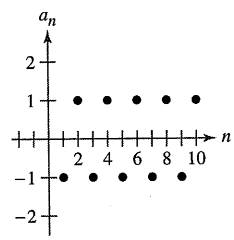


9. $a_n = \frac{10}{n+1}$
10. $a_n = \frac{10n}{n+1}$
11. $a_n = (-1)^n$
12. $a_n = \frac{(-1)^n}{n}$

Writing Terms In Exercises 13–16, write the next two apparent terms of the sequence. Describe the pattern you used to find these terms.

13. 2, 5, 8, 11, . . .
14. 8, 13, 18, 23, 28, . . .
15. 5, 10, 20, 40, . . .
16. $6, -2, \frac{2}{3}, -\frac{2}{9}, \ldots$

Simplifying Factorials In Exercises 17–20, simplify the ratio of factorials.

17. $\frac{(n+1)!}{n!}$
18. $\frac{n!}{(n+2)!}$
19. $\frac{(2n-1)!}{(2n+1)!}$
20. $\frac{(2n+2)!}{(2n)!}$

Finding the Limit of a Sequence In Exercises 21–24, find the limit (if possible) of the sequence.

21. $a_n = \frac{5n^2}{n^2+2}$
22. $a_n = 6 + \frac{2}{n^2}$
23. $a_n = \frac{2n}{\sqrt{n^2+1}}$
24. $a_n = \cos \frac{2}{n}$

Finding the Limit of a Sequence In Exercises 25–28, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

25. $a_n = \frac{4n+1}{n}$
26. $a_n = \frac{1}{n^{3/2}}$
27. $a_n = \sin \frac{n\pi}{2}$
28. $a_n = 2 - \frac{1}{4^n}$

Determining Convergence or Divergence In Exercises 29–44, determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

29. $a_n = \frac{5}{n+2}$
30. $a_n = 8 + \frac{5}{n}$
31. $a_n = (-1)^n\left(\frac{n}{n+1}\right)$
32. $a_n = \frac{1+(-1)^n}{n^2}$
33. $a_n = \frac{10n^2+3n+7}{2n^2-6}$
34. $a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$
35. $a_n = \frac{\ln(n^3)}{2n}$
36. $a_n = \frac{5^n}{3^n}$
37. $a_n = \frac{(n+1)!}{n!}$
38. $a_n = \frac{(n-2)!}{n!}$
39. $a_n = \frac{n^p}{e^n},\ p > 0$
40. $a_n = n \sin \frac{1}{n}$
41. $a_n = 2^{1/n}$
42. $a_n = -3^{-n}$
43. $a_n = \frac{\sin n}{n}$
44. $a_n = \frac{\cos \pi n}{n^2}$

Finding the nth Term of a Sequence In Exercises 45–52, write an expression for the nth term of the sequence. (There is more than one correct answer.)

45. 2, 8, 14, 20, . . .
46. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$
47. −2, 1, 6, 13, 22, . . .
48. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$
49. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
50. 2, 24, 720, 40,320, 3,628,800, . . .
51. $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$
52. $\frac{1}{2 \cdot 3}, \frac{2}{3 \cdot 4}, \frac{3}{4 \cdot 5}, \frac{4}{5 \cdot 6}, \ldots$

Finding Monotonic and Bounded Sequences **In Exercises 53–60, determine whether the sequence with the given nth term is monotonic and whether it is bounded. Use a graphing utility to confirm your results.**

53. $a_n = 4 - \dfrac{1}{n}$

54. $a_n = \dfrac{3n}{n+2}$

55. $a_n = ne^{-n/2}$

56. $a_n = \left(-\dfrac{2}{3}\right)^n$

57. $a_n = \left(\dfrac{2}{3}\right)^n$

58. $a_n = \left(\dfrac{3}{2}\right)^n$

59. $a_n = \sin \dfrac{n\pi}{6}$

60. $a_n = \dfrac{\cos n}{n}$

Using a Theorem **In Exercises 61–64, (a) use Theorem 9.5 to show that the sequence with the given nth term converges, and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.**

61. $a_n = 7 + \dfrac{1}{n}$

62. $a_n = 5 - \dfrac{2}{n}$

63. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

64. $a_n = 2 + \dfrac{1}{5^n}$

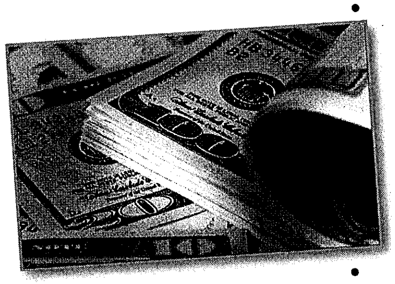
65. **Increasing Sequence** Let $\{a_n\}$ be an increasing sequence such that $2 \le a_n \le 4$. Explain why $\{a_n\}$ has a limit. What can you conclude about the limit?

66. **Monotonic Sequence** Let $\{a_n\}$ be a monotonic sequence such that $a_n \le 1$. Discuss the convergence of $\{a_n\}$. When $\{a_n\}$ converges, what can you conclude about its limit?

67. **Compound Interest**

Consider the sequence $\{A_n\}$ whose nth term is given by

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$



where P is the principal, A_n is the account balance after n months, and r is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence when $P = \$10{,}000$ and $r = 0.055$.

68. **Compound Interest** A deposit of \$100 is made in an account at the beginning of each month at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)(1.0025^n - 1)$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.

WRITING ABOUT CONCEPTS

69. **Sequence** Is it possible for a sequence to converge to two different numbers? If so, give an example. If not, explain why not.

70. **Defining Terms** In your own words, define each of the following.

(a) Sequence (b) Convergence of a sequence

(c) Monotonic sequence (d) Bounded sequence

71. **Writing a Sequence** Give an example of a sequence satisfying the condition or explain why no such sequence exists. (Examples are not unique.)

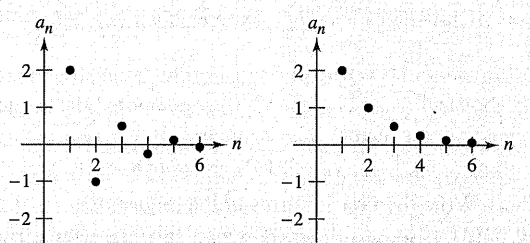
(a) A monotonically increasing sequence that converges to 10

(b) A monotonically increasing bounded sequence that does not converge

(c) A sequence that converges to $\frac{3}{4}$

(d) An unbounded sequence that converges to 100

72. **HOW DO YOU SEE IT?** The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.



73. **Government Expenditures** A government program that currently costs taxpayers \$4.5 billion per year is cut back by 20 percent per year.

(a) Write an expression for the amount budgeted for this program after n years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

74. **Inflation** When the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$25,000, the average price after n years is $P_n = \$25{,}000(1.045)^n$. Compute the average prices for the next 5 years.

75. **Using a Sequence** Compute the first six terms of the sequence $\{a_n\} = \{\sqrt[n]{n}\}$. If the sequence converges, find its limit.

76. **Using a Sequence** Compute the first six terms of the sequence

$$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}.$$

If the sequence converges, find its limit.

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77. **Proof** Prove that if $\{s_n\}$ converges to L and $L > 0$, then there exists a number N such that $s_n > 0$ for $n > N$.

78. **Modeling Data** The amounts of the federal debt a_n (in trillions of dollars) of the United States from 2000 through 2011 are given below as ordered pairs of the form (n, a_n), where n represents the year, with $n = 0$ corresponding to 2000. *(Source: U.S. Office of Management and Budget)*

$(0, 5.6)$, $(1, 5.8)$, $(2, 6.2)$, $(3, 6.8)$, $(4, 7.4)$, $(5, 7.9)$, $(6, 8.5)$, $(7, 9.0)$, $(8, 10.0)$, $(9, 11.9)$, $(10, 13.5)$, $(11, 14.8)$

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \quad n = 0, 1, \ldots, 11$$

for the data. Use the graphing utility to plot the points and graph the model.

(b) Use the model to predict the amount of the federal debt in the year 2020.

True or False? In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

79. If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

80. If $\{a_n\}$ converges, then $\lim_{n\to\infty} (a_n - a_{n+1}) = 0$.

81. If $\{a_n\}$ converges, then $\{a_n/n\}$ converges to 0.

82. If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

83. **Fibonacci Sequence** In a study of the progeny of rabbits, Fibonacci (ca. 1170–ca. 1240) encountered the sequence now bearing his name. The sequence is defined recursively as $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Write the first 12 terms of the sequence.

(b) Write the first 10 terms of the sequence defined by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

(c) Using the definition in part (b), show that

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

(d) The **golden ratio** ρ can be defined by $\lim_{n\to\infty} b_n = \rho$. Show that

$$\rho = 1 + \frac{1}{\rho}$$

and solve this equation for ρ.

84. **Using a Theorem** Show that the converse of Theorem 9.1 is not true. [*Hint:* Find a function $f(x)$ such that $f(n) = a_n$ converges, but $\lim_{x\to\infty} f(x)$ does not exist.]

85. **Using a Sequence** Consider the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim_{n\to\infty} a_n$.

86. **Using a Sequence** Consider the sequence $\{a_n\}$ where $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{k + a_n}$, and $k > 0$.

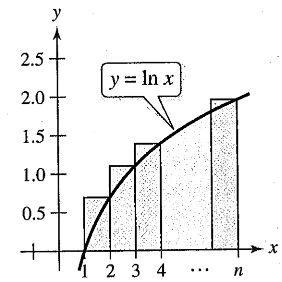
(a) Show that $\{a_n\}$ is increasing and bounded.

(b) Prove that $\lim_{n\to\infty} a_n$ exists.

(c) Find $\lim_{n\to\infty} a_n$.

87. **Squeeze Theorem**

(a) Show that $\int_1^n \ln x\, dx < \ln(n!)$ for $n \geq 2$.



(b) Draw a graph similar to the one above that shows

$$\ln(n!) < \int_1^{n+1} \ln x\, dx.$$

(c) Use the results of parts (a) and (b) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \text{ for } n > 1.$$

(d) Use the Squeeze Theorem for Sequences and the result of part (c) to show that $\lim_{n\to\infty} \left(\sqrt[n]{n!}/n\right) = 1/e$.

(e) Test the result of part (d) for $n = 20, 50$, and 100.

88. **Proof** Prove, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

89. **Proof** Prove, using the definition of the limit of a sequence, that $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

90. **Using a Sequence** Find a divergent sequence $\{a_n\}$ such that $\{a_{2n}\}$ converges.

91. **Proof** Prove Theorem 9.5 for a nonincreasing sequence.

PUTNAM EXAM CHALLENGE

92. Let $\{x_n\}$, $n \geq 0$, be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

93. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

2, 3, 6, 14, 40, 152, 784, 5168, 40,576

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

These problems were composed by the Committee on the Putnam Prize Competition.