Key

BC Calculus – 9.2 Notes – 2nd Derivative of Parametric Equations

Second Derivative of a Parametric Equation

The second derivative of a parametric given by $x = f(t)$ and $y = g(t)$ is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \rightarrow \frac{\left[\frac{dy}{dx}\right]'}{x'(t)}$$

Given the following parametric equations, find $\frac{d^2y}{dx^2}$ in terms of t.

1. $x(t) = \sqrt{t}$ and $y(t) = \frac{1}{2}(t^2 - 2)$ for $t \geq 0$.

$y = \frac{1}{2}t^2 - 1$

$$\frac{dy}{dx} = \frac{t}{\frac{1}{2}t^{-1/2}}$$

$$\frac{dy}{dx} = 2t^{3/2}$$

$$\frac{d^2y}{dx^2} = \frac{2 \cdot \frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} \leftarrow x'(t)$$

$$\frac{d^2y}{dx^2} = 6t^{1/2} \cdot t^{1/2} = \boxed{6t}$$

2. $x = 3\cos t$ and $y = 4\sin t$.

$$\frac{dy}{dx} = \frac{4\cos t}{-3\sin t} = -\frac{4}{3}\cot(t)$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{4}{3}\left(-\csc^2(t)\right)}{-3\sin(t)}$$

$$\frac{d^2y}{dx^2} = \frac{4}{3}\csc^2(t) \cdot \frac{-1}{3}\csc(t)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-4}{9}\csc^3(t)}$$

3. At $t = 1$, find the concavity of the graph defined parametrically by $x = t^3 + 1$ and $y = t^4 + t$.

2nd derivative

$$\frac{dy}{dx} = \frac{4t^3 + 1}{3t^2} \rightarrow \frac{4t^3}{3t^2} + \frac{1}{3t^2}$$

$$\frac{dy}{dx} = \frac{4}{3}t + \frac{1}{3}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{4}{3} - \frac{2}{3}t^{-3}}{3t^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{\frac{4}{3} - \frac{2}{3(1)^3}}{3(1)^2} \rightarrow \frac{\frac{4}{3} - \frac{2}{3}}{3} \rightarrow \frac{\frac{2}{3}}{3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} > 0$$

$\boxed{\text{Graph is concave up at } t=1}$

9.2 practice problems

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'}{dx/dt}$$

Given the following parametric equations, find $\frac{d^2y}{dx^2}$ in terms of t.

1. $x(t) = e^{-2t}$ and $y(t) = e^{2t}$.

$$\frac{dy}{dx} = \frac{2e^{2t}}{-2e^{-2t}} = -e^{+4t}$$

$$\frac{d^2y}{dx^2} = \frac{-4e^{+4t}}{-2e^{-2t}} = \boxed{2e^{6t}}$$

2. $x(t) = t^3$ and $y(t) = t^4 + 1$ for $t > 0$.

$$\frac{dy}{dx} = \frac{4t^3}{3t^2} = \frac{4}{3}t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{4}{3}}{3t^2} = \frac{4}{3}\cdot\frac{1}{3t^2} = \boxed{\frac{4}{9t^2}}$$

3. $x(t) = at^3$ and $y(t) = bt$, where a and b are positive constants.

$$\frac{dy}{dx} = \frac{b}{3at^2} \rightarrow \frac{b}{3a}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-2bt^{-3}}{3a}}{3at^2} \rightarrow \frac{-2b}{3at^3}\cdot\frac{1}{3at^2} = \boxed{\frac{-2b}{9a^2t^5}}$$

4. $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = \sin(t^2)$.

$$\frac{dy}{dx} = \frac{\sin(t^2)}{4} = \frac{1}{4}\sin(t^2)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{4}\cos(t^2)(2t)}{4} \rightarrow \frac{1}{2}t\cos(t^2)\cdot\frac{1}{4}$$

$$= \boxed{\frac{1}{8}t\cos(t^2)}$$

5. $x = e^t$ and $y = te^{-t}$.

$$\frac{dy}{dx} = \frac{1e^{-t} + te^{-t}(-1)}{e^t} = \frac{e^{-t}(1-t)}{e^t}$$

$$= \frac{1-t}{e^{2t}} = e^{-2t}(1-t)$$

$$\frac{d^2y}{dx^2} = \frac{-2e^{-2t}(1-t) + e^{-2t}(-1)}{e^t}$$

$$= e^{-2t}[-2+2t-1]\cdot e^{-t}$$

$$= \boxed{e^{-3t}(2t-3)} \text{ or } \boxed{\frac{2t-3}{e^{3t}}}$$

6. $x = t^2 + 1$ and $y = 2t^3$.

$$\frac{dy}{dx} = \frac{6t^2}{2t} \rightarrow 3t$$

$$\frac{d^2y}{dx^2} = \boxed{\frac{3}{2t}}$$

7. Given a curve defined by the parametric equations $x(t) = 2 - t^2$ and $y(t) = t^2 + t^3$. Determine the open t-intervals on which the curve is concave up or down.

$$\frac{dy}{dx} = \frac{2t+3t^2}{-2t} = \frac{2t}{-2t} + \frac{3t^2}{-2t} = -1 - \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{-3/2}{-2t} = \frac{+3}{4t} \quad \text{critical point } t=0$$

y'' : $-$ (∩) | $+$ (∪) at 0

$$\boxed{\begin{array}{l}\text{Concave up } (0,\infty)\\ \text{concave down } (-\infty,0)\end{array}}$$

8. If $x(\theta) = 2 + \sec\theta$ and $y(\theta) = 1 + 2\tan\theta$, Find the slope and the concavity at $\theta = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} \rightarrow \frac{2\sec\theta}{\tan\theta} \rightarrow 2\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{2}{\sin\theta} = 2\csc\theta \qquad \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = 2\csc\pi/6 = 2\left(\frac{2}{1}\right)$$

slope $\rightarrow = \boxed{4}$

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} \rightarrow \frac{-2\cos\theta}{\sin\theta}\cdot\cot\theta\cdot\cot\theta$$

$$\frac{d^2y}{dx^2} = -2\cot^3\theta$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/6} = -2\left[\cot(\pi/6)\right]^3 = -2\left(\frac{\sqrt{3}}{1}\right)^3 = -2\cdot 3\sqrt{3}$$

$$\boxed{= -6\sqrt{3} \text{ (concave down)}}$$

9. If $x = \cos\theta$ and $y = 3\sin\theta$, find the slope and concavity at $\theta = 0$.

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = -3\cot(0) = -3\left(\frac{1}{0}\right) \text{ undefined slope}$$

$$\frac{d^2y}{dx^2} = \frac{+3\csc^2\theta}{-\sin\theta} = -3\csc^3\theta \text{ or } \frac{-3}{\sin^3\theta}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=0} = \frac{-3}{0} \leftarrow \text{undefined concavity}$$

10. If $x(t) = t - \ln t$ and $y(t) = t + \ln t$, determine values of t where the graph is concave up.

$$\frac{dy}{dx} = \frac{1 + 1/t}{1 - 1/t} \rightarrow \frac{\frac{t+1}{t}}{\frac{t-1}{t}} \rightarrow \frac{t+1}{\cancel{t}}\cdot\frac{\cancel{t}}{t-1}$$

$$\frac{dy}{dx} = \frac{t+1}{t-1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1(t-1)-(t+1)(1)}{(t-1)^2}}{\frac{t-1}{t}} \rightarrow \frac{\frac{t-1-t-1}{(t-1)^2}}{\frac{t-1}{t}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-2}{(t-1)^2}}{\frac{t-1}{t}} \rightarrow \frac{-2}{(t-1)^2}\cdot\frac{t}{t-1} \rightarrow \frac{-2t}{(t-1)^3}$$

critical pts: $t=0, t=1$

y'' : $-$ (∩) at -1 | 0 | $+$ (∪) at $1/2$ | 1 | $-$ (∩) at 2

$$\boxed{f(x) \text{ is concave up on } (0,1)}$$

9.2 Second Derivatives of Parametric Equations

Test Prep

11. If $x = 3t^2 - 1$ and $y = \ln t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

$$\frac{dy}{dx} = \frac{\frac{1}{t}}{6t} \rightarrow \frac{1}{t} \cdot \frac{1}{6t} = \frac{1}{6t^2} = \frac{1}{6}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-2}{6}t^{-3}}{6t} \rightarrow \frac{-1}{3t^3} \cdot \frac{1}{6t} = \boxed{\frac{-1}{18t^4} = \frac{-1}{18}t^{-4}}$$

A. $\frac{1}{6}t^2$ B. $-\frac{1}{3}t^{-3}$ C. $\boxed{-\frac{1}{18}t^{-4}}$ D. $-\frac{1}{2}t^{-4}$ E. $6t^4$

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12. If $x = \theta - \cos\theta$ and $y = 1 - \sin\theta$, find the slope and concavity at $\theta = \pi$.

$$\frac{dy}{dx} = \frac{-\cos\theta}{1-(-\sin\theta)} = \frac{-\cos\theta}{1+\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-\cos\pi}{1+\sin\pi} = \frac{-(-1)}{1+0} = \boxed{1}$$ slope

$$\frac{d^2y}{dx^2} = \frac{\frac{\sin\theta(1+\sin\theta)-(-\cos\theta)(\cos\theta)}{(1+\sin\theta)^2}}{1+\sin\theta} \rightarrow \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{(1+\sin\theta)^2} \cdot \frac{1}{1+\sin\theta} = \frac{\sin\theta + 1}{(1+\sin\theta)^3}$$

$$\frac{d^2y}{dx} = \frac{1}{(1+\sin\theta)^2} \quad \left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{1}{(1+\sin\pi)^2} = \frac{1}{1} = \boxed{1}$$ concave up

A. Slope: -1, Concave down B. Slope: π, Concave up C. Slope: 1, Concave down

$\boxed{\text{D. Slope: 1, Concave up}}$ E. Slope: $\frac{1}{\pi}$, Concave up