Key

9.2 AP Practice Problems (p.660) – Parametric Arc Length

1. The smooth curve C is represented by the parametric equations $x(t) = \tan t$, $y(t) = t^2 - 3t + 8$, $-\frac{\pi}{4} \le t < \frac{\pi}{2}$. The slope of the tangent line to C at the point (0, 8) is

$\boxed{\text{(A)}}$ -3 (B) $-\frac{3}{2}$ (C) 0 (D) $-\frac{1}{3}$

x y

$x = \tan(t)$

$0 = \tan(t)$

$t = 0, \frac{\pi}{2}, -\frac{\pi}{2}$

$y = t^2 - 3t + 8$

$8 = t^2 - 3t + 8$

$0 = t^2 - 3t$

$0 = t(t-3)$

$t = 0,\ t = 3$

$dx/dt = \sec^2(t)$

$dy/dt = 2t - 3$

$\frac{dy/dt}{dx/dt} \rightarrow \frac{2t-3}{\sec^2(t)}$

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{2(0)-3}{(\sec(0))^2} = \frac{-3}{1} = -3$

$\boxed{\frac{dy}{dx} = -3}$

2. Which of the following is an equation of the tangent line to the plane curve represented by the parametric equations $x(t) = 2t - 5$, $y(t) = t^3$ when $t = 2$?

(A) $y = -6x + 14$ (B) $y = 6x$

(C) $y = 6x + 2$ (D) $y = 6x + 14$

$x(2) = 4 - 5 = -1$

$y(2) = 2^3 = 8$

$\frac{dy}{dt} = 3t^2$

$dx/dt = 2$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3t^2}{2} \rightarrow \frac{3(2)^2}{2} = 6$

point: $(-1, 8)$

slope: $m = 6$

$y - y_1 = m(x - x_1)$

$y - 8 = 6(x - -1)$

$y - 8 = 6(x + 1)$

$y = 6x + 6 + 8$

$\boxed{y = 6x + 14}$

3. For the smooth curve, $x(t) = t^3 - 12t$, $y(t) = 4t^2 + t$, find all the points where the tangent line is either horizontal or vertical.

(A) Horizontal: $t = \frac{1}{8}$; Vertical: $t = -2, 2$

(B) Horizontal: $t = -2, 2$; Vertical: none

$\boxed{\text{(C)}}$ Horizontal: $t = -\frac{1}{8}$; Vertical: $t = -2, 2$

(D) Horizontal: none, Vertical $t = -\frac{1}{8}$

horizontal tangent when $\frac{dy}{dt} = 0$

$y = 4t^2 + t$

$dy/dt = 8t + 1$

$0 = 8t + 1$

$t = -1/8$

vertical tangent when $dx/dt = 0$

$\frac{dx}{dt} = 3t^2 - 12$

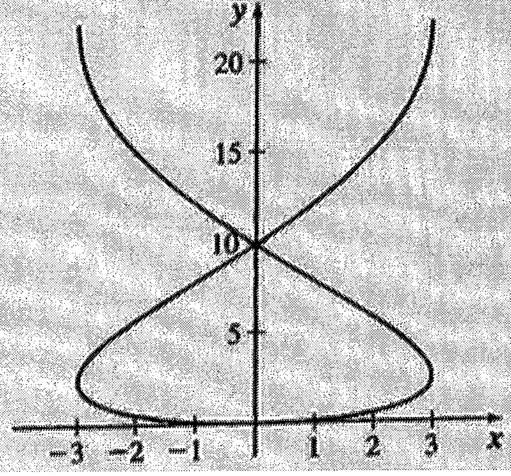
$0 = 3(t^2 - 4)$

$t = 2,\ t = -2$

4. The plane curve represented by the parametric equations

$$x(t) = 3\sin t,\ y(t) = t^2,\ -\frac{3}{2}\pi \le t \le \frac{3}{2}\pi$$

is shown below. The graph intersects itself at the point $(0, \pi^2)$.



(a) Find the numbers t that correspond to the point $(0, \pi^2)$.

(b) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and confirm that $\frac{dx}{dt} \neq 0$ at the point of intersection.

(c) Find the slope of the tangent lines at the point of intersection.

(d) Find an equation of each tangent line at the point of intersection.

a) $x = 3\sin(t)$ | $y = t^2$

$(0, \pi^2)$ → $0 = 3\sin(t)$ | $\pi^2 = t^2$

x y

$t = 0, \pi, 2\pi, \dots, -\pi$ | $t = \pm\pi$

a) $\boxed{t = \pi,\ t = -\pi}$ since $-\frac{3}{2}\pi \le t \le \frac{3\pi}{2}$

b) $\frac{dx}{dt} = 3\cos(t)$ $\quad \frac{dy}{dt} = 2t$

$\left.\frac{dx}{dt}\right|_{t=\pi} = 3\cos(t) \to 3\cos(\pi) = -3 \neq 0$ ✓

$\left.\frac{dx}{dt}\right|_{t=-\pi} = 3\cos(t) \to 3\cos(-\pi) = -3 \neq 0$ ✓

c) $\left.\frac{dy/dt}{dx/dt}\right|_{t=\pi} = \frac{2t}{3\cos(t)} = \frac{2\pi}{3\cos\pi} = \boxed{\frac{-2\pi}{3}}$

$\left.\frac{dy}{dx}\right|_{t=-\pi} = \frac{-2\pi}{3\cos(-\pi)} = \frac{-2\pi}{-3} = \boxed{\frac{2\pi}{3}}$

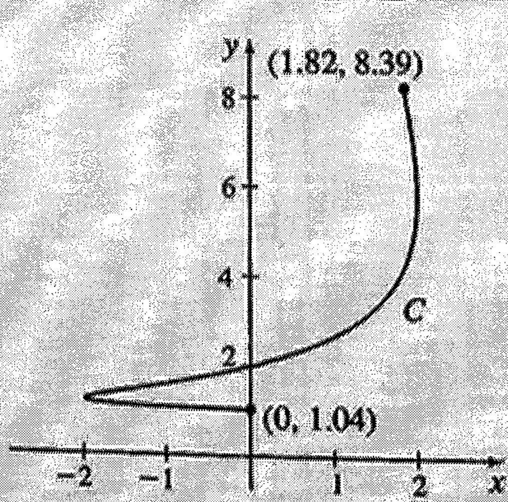
d) point: $(0, \pi^2)$ | point $(0, \pi^2)$

slope: $m = -2\pi/3$ | slope: $m = 2\pi/3$

$y - \pi^2 = -\frac{2\pi}{3}(x - 0)$ | $y - \pi^2 = \frac{2\pi}{3}(x - 0)$

$\boxed{y = -\frac{2\pi}{3}x + \pi^2}$ | $\boxed{y = \frac{2\pi}{3}x + \pi^2}$

5. The smooth curve C represented by the parametric equations $x(t) = 2\sin t$, $y(t) = 1 + e^t$, $-\pi \le t \le 2$, is shown below.



The arc length of C is given by the integral

(A) $\int_{-2}^{1.8} \sqrt{4\cos^2 t + e^{2t}}\, dt$ (B) $\int_{-\pi}^{2} \sqrt{4\sin^2 t + (1 + e^t)^2}\, dt$

(C) $\int_{-\pi}^{2} \sqrt{4\cos^2 t + e^{2t}}\, dt$ (D) $\int_{0}^{1.8} \sqrt{4\cos^2 t + e^{2t}}\, dt$

$* L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dx}{dt} = 2\cos(t)$ $\frac{dy}{dt} = e^t$

$L = \int_{-\pi}^{2} \sqrt{[2\cos(t)]^2 + [e^t]^2}\, dt$

$\boxed{L = \int_{-\pi}^{2} \sqrt{4\cos^2(t) + e^{2t}}\, dt}$

6. The length of the curve represented by the parametric equations $x(t) = 2 + 4t^3$, $y(t) = 1 + 6t^2$, from $t = 0$ to $t = 1$ is

(A) $8\sqrt{2} - 4$ (B) $2^{7/2} - 1$ (C) 4 (D) $16\sqrt{2} - 8$

$\frac{dx}{dt} = 12t^2$ $\frac{dy}{dt} = 12t$

$L = \int_0^1 \sqrt{(12t^2)^2 + (12t)^2}\, dt$

$\int \sqrt{144t^4 + 144t^2}$

$\sqrt{144t^2(t^2+1)}$

$\int 12t\sqrt{t^2+1}\, dt$

$*$ u-sub

$u = t^2 + 1$

$\frac{du}{dt} = 2t$

$dt = \frac{du}{2t}$

$\int 12t \cdot u^{1/2} \cdot \frac{du}{2t}$

$6\int u^{1/2} du \to 6 \cdot \frac{u^{3/2}}{3/2}$

$6 \cdot \frac{2}{3} u^{3/2}$

$4(t^2+1)^{3/2}\Big]_0^1$

$4(2)^{3/2} - 4(1)^{3/2}$

$4\sqrt{8} - 4(1)$

$4 \cdot \sqrt{4 \cdot 2} - 4$

$\boxed{8\sqrt{2} - 4}$

7. **An object moves along the plane curve C represented by the parametric equations $x(t) = \cos(2t)$, $y(t) = \cos^2 t$. The distance the object travels from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{3}$ is**

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{5}}{4}$ (C) $\frac{5}{4}$ (D) $\frac{\sqrt{5}}{2}$

$x'(t) = -\sin(2t) \cdot 2$

$y(t) = (\cos t)^2$

$y'(t) = 2(\cos(t)) \cdot -\sin(t)$

$y'(t) = -\sin(2t)$

Recall identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$L = \int_{\pi/4}^{\pi/3} \sqrt{(-2\sin(2t))^2 + (-\sin(2t))^2}\, dt$$

$$L = \int \sqrt{4\sin^2(2t) + \sin^2(2t)}\, dt$$

$$= \int \sqrt{5\sin^2(2t)}\, dt$$

$$= \int_{\pi/4}^{\pi/3} \sqrt{5} \cdot \sin(2t)\, dt$$

$u = 2t$ | $dt = \frac{du}{2}$

$\frac{du}{dt} = 2$

$$\int \sqrt{5} \sin u \cdot \frac{du}{2}$$

$$\frac{\sqrt{5}}{2} \int \sin u\, du$$

$$\frac{\sqrt{5}}{2} \cdot -\cos u$$

$$\left.\frac{\sqrt{5}}{2}\cos(2t)\right]_{\pi/4}^{\pi/3}$$

$$\frac{-\sqrt{5}}{2}\cos\frac{2\pi}{3} - -\frac{5}{2}\cos\left(\frac{\pi}{2}\right)$$

$$\frac{-\sqrt{5}}{2} \cdot \left(\frac{-1}{2}\right) + \frac{5}{2}(0) = \boxed{\frac{\sqrt{5}}{4}}$$