Calculus BC Ch. 9.2 Notes Series and Convergence

-Understand the definition of a convergent infinite series
-Use properties of **infinite geometric series**
-Use the **nth-Term Test for Divergence** of an infinite series

One important application of infinite sequences is in representing infinite summations. If $\{a_n\}$ is an infinite sequence, then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ \dots\ + a_n + \ \dots$ is an **infinite series** , simply the sum of the terms in a sequence. The numbers a_1, a_2, a_3, are the **terms** of the series. Unlike sequences which always start at 1, sometimes it is convenient to begin the index at n = 0 (or some other integer).
To find the sum of an infinite series, consider the following **sequence of partial sums:**

$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3 \qquad S_n = a_1 + a_2 + a_3 + \dots + a_n$

If this sequence of partial sums converges, the series is said to converge.

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$ the **nth partial sum** is given by $S_n = a_1 + a_2 + a_3 + \dots + a_n$. If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges**. The limit S is called the **sum of the series.**

If $\{S_n\}$ diverges, then the series **diverges.**

Example) Convergent and Divergent Series Write a few partial sums and determine if the series converges.

1) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

2) $\sum_{n=1}^{\infty} 1$

3) $\sum_{n=1}^{\infty} \frac{1}{n}$

4) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

A series such as $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$ is called a **telescoping series** because the series collapses to one term or just a few terms. If a telescoping series converges, it will converge to $S = b_1 - \lim_{n\to\infty} b_{n+1}$ by the Telescoping Series Test

5) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

6) Writing a Series in Telescoping Form

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$

Geometric Series

Series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ... + ar^n + ..., a \neq 0$ are called a **geometric series** with ratio r.

A geometric series with ratio *r* diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum given by:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Example) Convergent and Divergent Geometric Series

7) $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

8) $\sum_{n=0}^{\infty} \frac{3}{2^n}$

The next few sections of this chapter are concerned with determining whether a function converges or diverges. There are several tests for convergence/divergence that you will have to know. The first is this:

nth-Term Test for Divergence

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. ***The converse of this is not true. If $\lim_{n \to \infty} a_n = 0$ the test is inconclusive***

Examples: Determine if the Series Diverges

9) $\sum_{n=0}^{\infty} 2^n$

10) $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$

11) $\sum_{n=1}^{\infty} \frac{1}{n}$

12) $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$

13) $\sum_{n=2}^{\infty} \frac{1}{(1.1)^n}$

Key

Calculus BC 9.2 Notes Series and Convergence

-Understand the definition of a convergent infinite series
-Use properties of **infinite geometric series**
-Use the **nth-Term Test for Divergence** of an infinite series

*It is one thing to see whether the terms of a sequence converges or diverges. It's another thing entirely to see if the sum of the terms converge.

One important application of infinite sequences is in representing infinite summations. If $\{a_n\}$ is an infinite sequence, then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$ is an **infinite series**, simply the sum of the terms in a sequence. The numbers a_1, a_2, a_3, are the **terms** of the series. Unlike sequences which always start at 1, sometimes it is convenient to begin the index at n = 0 (or some other integer).

To find the sum of an infinite series, consider the following **sequence of partial sums:**

$S_1 = a_1 \quad S_2 = a_1 + a_2 \quad S_3 = a_1 + a_2 + a_3 \quad S_n = a_1 + a_2 + a_3 + \dots + a_n$ S_n is the n^{th} partial sum.

* If this sequence of partial sums converges, the series is said to converge.

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$ the **nth partial sum** is given by $S_n = a_1 + a_2 + a_3 + \dots + a_n$. If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges**. The limit S is called the **sum of the series**.

$S_1, S_2, S_3, S_4, \dots$

If $\{S_n\}$ diverges, then the series **diverges**.

Example) Convergent and Divergent Series Write a few partial sums and determine if the series converges.

1) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = 3/4$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 7/8$

convergent series

2) $\sum_{n=1}^{\infty} 1$ $1+1+1+1+\dots$ divergent series

$S_1 = 1$

$S_2 = 1+1$

$S_3 = 1+1+1$

3) $\sum_{n=1}^{\infty} \frac{1}{n}$ $a_n = \frac{1}{n}$ $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2} \approx 1.5$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \approx 1.83$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.08$

*This sum is growing, but is the sum going to taper off towards a specific value? or get infinitely large?

sum(seq(1/x, x, 1, 999)) = 7.484 $S_{999} = 7.484$, Sum diverges to ∞

4) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$S_1 = 1$

$S_2 = 1 + \frac{1}{4}$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9}$

$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \approx S_{999} \approx 1.6$

Sum converges.

A series such as $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$ is called a **telescoping series** because the series collapses to one term or just a few terms. If a telescoping series converges, it will converge to $S = b_1 - \lim_{n\to\infty} b_{n+1}$ by the Telescoping Series Test

* 2 ratios, usually subtracted and similar to each other.
These ratios differ by only a couple of values so that it will generate the same values but with different signs where down the line they will cancel out.

5) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots$

$= \boxed{1}$

Sum converges to 1.

6) Writing a Series in Telescoping Form

Find the sum of the series $\sum_{n=1}^{\infty}\frac{2}{4n^2-1} = \frac{2}{(2n+1)(2n-1)} = \sum_{n=1}^{\infty}\left[\frac{-1}{2n+1}+\frac{1}{2n-1}\right]$

$= \left(-\frac{1}{3}+1\right)+\left(\frac{-1}{5}+\frac{1}{3}\right)+\left(-\frac{1}{7}+\frac{1}{5}\right)+\ldots$

$= 1$ The sum converges to 1.

Geometric Series (GST)

Series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots, a \neq 0$ are called a **geometric series** with ratio r.

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum given by: $\sum_{n=0}^{\infty} ar^n = \frac{a_1}{1-r}, \quad 0<|r|<1.$ $S = \frac{a_1}{1-r}$

* One of the few tests where we can find the sum once we determine that the series converges.

Example) Convergent and Divergent Geometric Series

7) $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$

$|r| = \frac{3}{2} \geq 1$, so series diverges

8) $\sum_{n=0}^{\infty}\frac{3}{2^n} = \sum_{n=1}^{\infty} 3\left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n$, $|r| = \left|\frac{1}{2}\right| < 1$, so series converges

$S = \frac{a_1}{1-r} = \frac{3/2}{1-1/2} = \frac{3/2}{1/2}$

Series converges to 3.

The next few sections of this chapter are concerned with determining whether a function converges or diverges. There are several tests for convergence/divergence that you will have to know. The first is this:

nth-Term Test for Divergence * Does not test for convergence!!

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. ***The converse of this is not true. If $\lim_{n\to\infty} a_n = 0$ the test is inconclusive***

* This is only a 1-way test. * This should always be the first test we check, as this can be a time saver.

Examples: Determine if the Series Diverges

9) $\sum_{n=0}^{\infty} 2^n$

Since $\lim_{n\to\infty} 2^n \neq 0$ then $\sum_{n=1}^{\infty} 2^n$ diverges. by the n^{th} term test

10) $\sum_{n=1}^{\infty}\frac{n!}{2n!+1}$ $\lim_{n\to\infty}\frac{n!}{2n!+1} = \frac{1}{2} \neq 0$

Since $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. (by the n^{th} term test)

11) $\sum_{n=1}^{\infty}\frac{1}{n}$

$\lim_{n\to\infty}\frac{1}{n} = 0$

Inconclusive test for convergence.

12) $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$

Since $\lim_{n\to\infty} \frac{2n+3}{3n-5} = \frac{2}{3} \neq 0$,

series diverges by n^{th} term test.

* Though the sequence converges to $\frac{2}{3}$, but the series diverge (to $+\infty$)

13) $\sum_{n=2}^{\infty} \frac{1}{(1.1)^n}$

$\lim_{n\to\infty} \frac{1}{(1.1)^n} = 0$, n^{th} term test is inconclusive

$$\frac{1}{(1.1)^n} = \frac{1}{\left(1\frac{1}{10}\right)^n} = \frac{1}{\left(\frac{11}{10}\right)^n} = \left(\frac{10}{11}\right)^n$$

$\sum_{n=2}^{\infty} \left(\frac{10}{11}\right)^n$, $|r| = \frac{10}{11} < 1$, converges by GST. Series converges to

$$\frac{\left(\frac{10}{11}\right)^2}{1-\frac{10}{11}} = \frac{\frac{100}{121}}{\frac{1}{11}} = \boxed{\frac{100}{11}}$$