**93.** $T_n = n! + 2^n$

Use mathematical induction to verify the formula.

$T_0 = 1 + 1 = 2$

$T_1 = 1 + 2 = 3$

$T_2 = 2 + 4 = 6$

Assume $T_k = k! + 2^k$. Then

$$\begin{aligned} T_{k+1} &= (k + 1 + 4)T_k - 4(k + 1)T_{k-1} + (4(k + 1) - 8)T_{k-2} \\ &= (k + 5)\left[k! + 2^k\right] - 4(k + 1)\left((k - 1)! + 2^{k-1}\right) + (4k - 4)\left((k - 2)! + 2^{k-2}\right) \\ &= \left[(k + 5)(k)(k - 1) - 4(k + 1)(k - 1) + 4(k - 1)\right](k - 2)! + \left[(k + 5)4 - 8(k + 1) + 4(k - 1)\right]2^{k-2} \\ &= \left[k^2 + 5k - 4k - 4 + 4\right](k - 1)! + 8 \cdot 2^{k-2} \\ &= (k + 1)! + 2^{k+1}. \end{aligned}$$

By mathematical induction, the formula is valid for all $n$.

## Section 9.2 Series and Convergence

**1.** $S_1 = 1$

$S_2 = 1 + \frac{1}{4} = 1.2500$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$

**2.** $S_1 = \frac{1}{6} \approx 0.1667$

$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$

$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$

$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$

$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$

**3.** $S_1 = 3$

$S_2 = 3 - \frac{9}{2} = -1.5$

$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$

$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$

$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$

**4.** $S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{23}{12}$

$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{49}{24}$

**5.** $S_1 = 3$

$S_2 = 3 + \frac{3}{2} = 4.5$

$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$

$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$

$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$

**6.** $S_1 = 1$

$S_2 = 1 - \frac{1}{2} = 0.5$

$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$

$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$

$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$

**7.** $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$

Geometric series

$r = \frac{7}{6} > 1$

Diverges by Theorem 9.6

**8.** $\sum_{n=0}^{\infty} 4(-1.05)^n$

Geometric series

$|r| = |-1.05| = 1.05 > 1$

Diverges by Theorem 9.6

9. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 9.9

10. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

11. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 9.9

12. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 9.9

13. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}} = \lim_{n\to\infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

14. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 9.9

15. $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$

Geometric series with $r = \frac{5}{6} < 1$

Converges by Theorem 9.6

16. $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$

Geometric series with $|r| = \left|-\frac{1}{2}\right| < 1$

Converges by Theorem 9.6

17. $\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$

Converges by Theorem 9.6

18. $\sum_{n=0}^{\infty} (-0.6)^n$

Geometric series with $|r| = |-0.6| < 1$

Converges by Theorem 9.6

19. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots, \quad S_n = 1 - \frac{1}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

20. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

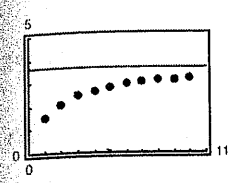
21. (a) $\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right) = 2\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right]$

$\left(S_n = 2\left[1 + \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)\right]\right) = 2\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{3} \approx 3.667$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 2.7976 | 3.1643 | 3.3936 | 3.5513 | 3.6078 |

(c)




(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

22. (a) $\sum_{n=1}^{\infty} \frac{4}{n(n+4)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

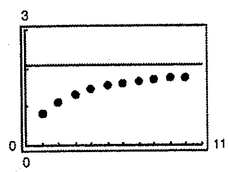
$$= \left(1 - \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.5377 | 1.7607 | 1.9051 | 2.0071 | 2.0443 |

(c)



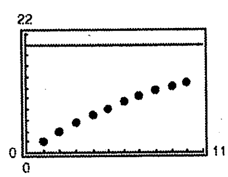

(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

23. (a) $\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 8.1902 | 13.0264 | 17.5685 | 19.8969 | 19.9995 |

(c)



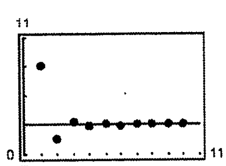

(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

24. (a) $\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1} = \sum_{n=0}^{\infty} 10\left(-\frac{1}{4}\right)^n = \frac{10}{1-(-1/4)} = 8$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 8.0078 | 7.99999 | 8.0000 | 8.0000 | 8.0000 |

(c)




(d) The terms of the series decrease in magnitude rapidly. So, the sequence of partial sums approaches the sum rapidly.

**25.** $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n = \frac{5}{1-(2/3)} = 15$

**26.** $\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n = \frac{1}{1-(-1/5)} = \frac{5}{6}$

**27.** $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$S_n = 2\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right] = 2\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$

$\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = 3$

**28.** $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$

$S_n = \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)\right] = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right)$

$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n+3}\right) = \frac{1}{6}$

**29.** $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-(3/4)} = 32$

**30.** $\sum_{n=0}^{\infty} 9\left(-\frac{1}{3}\right)^n = \frac{9}{1-(-1/3)} = \frac{27}{4}$

**31.** $\sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$

**32.** $\sum_{n=0}^{\infty}\left[(0.3)^n + (0.8)^n\right] = \sum_{n=0}^{\infty}\left(\frac{3}{10}\right)^n + \sum_{n=0}^{\infty}\left(\frac{8}{10}\right)^n$

$= \frac{1}{1-(3/10)} + \frac{1}{1-(8/10)}$

$= \frac{10}{7} + 5 = \frac{45}{7}$

**33.** Note that $\sin(1) \approx 0.8415 < 1$. The series $\sum_{n=1}^{\infty}[\sin(1)]^n$ is geometric with $r = \sin(1) < 1$. So,

$\sum_{n=1}^{\infty}[\sin(1)]^n = \sin(1)\sum_{n=0}^{\infty}[\sin(1)]^n = \frac{\sin(1)}{1-\sin(1)} \approx 5.3080$

**34.** $S_n = \sum_{k=1}^{n}\frac{1}{9k^2+3k-2} = \sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}$

$= \sum_{k=1}^{n}\left[\frac{1}{9k-3} - \frac{1}{9k+6}\right] = \frac{1}{3}\sum_{k=1}^{n}\left[\frac{1}{3k-1} - \frac{1}{3k+2}\right]$

$= \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) + \cdots + \left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)\right] = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6}$

**35.** (a) $0.\overline{4} = \sum_{n=0}^{\infty} \frac{4}{10}\left(\frac{1}{10}\right)^n$

(b) Geometric series with $a = \frac{4}{10}$ and $r = \frac{1}{10}$

$$S = \frac{a}{1-r} = \frac{4/10}{1-(1/10)} = \frac{4}{9}$$

**36.** (a) $0.\overline{36} = \sum_{n=0}^{\infty} \frac{36}{100}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{36}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{36/100}{1-(1/100)} = \frac{36}{99} = \frac{4}{11}$$

**37.** (a) $0.\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$$

**38.** (a) $0.\overline{01} = \sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = \frac{1}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n$

(b) $0.\overline{01} = \frac{1}{100}\cdot\frac{1}{1-(1/100)} = \frac{1}{100}\cdot\frac{100}{99} = \frac{1}{99}$

**39.** (a) $0.0\overline{75} = \sum_{n=0}^{\infty} \frac{3}{40}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{40}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{3/40}{99/100} = \frac{5}{66}$$

**40.** (a) $0.2\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200}\left(\frac{1}{100}\right)^n$

(b) Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

**41.** $\sum_{n=0}^{\infty}(1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 9.6

**42.** $\sum_{n=1}^{\infty}\frac{3^n}{1000}$

Geometric series with $r = 3 > 1$.

Diverges by Theorem 9.6

**43.** $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

$\lim_{n\to\infty}\frac{n+10}{10n+1} = \frac{1}{10} \neq 0$

Diverges by Theorem 9.9

**44.** $\sum_{n=1}^{\infty}\frac{4n+1}{3n-1}$

$\lim_{n\to\infty}\frac{4n+1}{3n-1} = \frac{4}{3} \neq 0$

Diverges by Theorem 9.9

**45.** $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_n = \left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1}-\frac{1}{n+1}\right) + \left(\frac{1}{n}-\frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{3}{2}, \text{ converges}$$

**46.** $\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$$S_n = \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{1}{2}, \text{ converges}$$

**47.** $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2)3^n}{3n^2}$$
$$= \lim_{n\to\infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n\to\infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule); diverges by Theorem 9.9

**48.** $\sum_{n=0}^{\infty} \frac{3}{5^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$, convergent

Geometric series with $r = \frac{1}{5}$

**49.** Because $n > \ln(n)$, the terms $a_n = \frac{n}{\ln(n)}$ do not approach 0 as $n \to \infty$. So, the series $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$ diverges.

**50.** $S_n = \sum_{k=1}^{n} \ln\left(\frac{1}{k}\right) = \sum_{k=1}^{n} -\ln(k)$

$= 0 - \ln 2 - \ln 3 - \cdots - \ln(n)$

Because $\lim_{n\to\infty} S_n$ diverges, $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ diverges.

**51.** For $k \neq 0$,

$$\lim_{n\to\infty}\left(1 + \frac{k}{n}\right)^n = \lim_{n\to\infty}\left[\left(1 + \frac{k}{n}\right)^{n/k}\right]^k$$
$$= e^k \neq 0.$$

For $k = 0, \lim_{n\to\infty} (1 + 0)^n = 1 \neq 0.$

So, $\sum_{n=1}^{\infty}\left[1 + \frac{k}{n}\right]^n$ diverges.

**52.** $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ converges because it is geometric with

$|r| = \frac{1}{e} < 1.$

**53.** $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

So, $\sum_{n=1}^{\infty} \arctan n$ diverges.

**54.** $S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right)$

$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right)$

$= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n)$

$= \ln(n+1) - \ln(1) = \ln(n+1)$

Diverges

**55.** See definitions on page 595.

**56.** $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

**57.** The series given by

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$

is a geometric series with ratio $r$. When $0 < |r| < 1$, the series converges to $a/(1 - r)$. The series diverges if $|r| \geq 1$.

**58.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

**59.** (a) $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

(b) $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

These are the same. The third series is different, unless $a_1 = a_2 = \cdots = a$ is constant.

(c) $\sum_{n=1}^{\infty} a_k = a_k + a_k + \cdots$

**60.** (a) Yes, the new series will still diverge.
(b) Yes, the new series will converge.

**61.** $\sum_{n=1}^{\infty} (3x)^n = (3x)\sum_{n=0}^{\infty} (3x)^n$

Geometric series: converges for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$f(x) = (3x)\sum_{n=0}^{\infty} (3x)^n = (3x)\frac{1}{1 - 3x} = \frac{3x}{1 - 3x}, \ |x| < \frac{1}{3}$

**62.** $\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$

Geometric series: converges for

$\left|\frac{2}{x}\right| < 1 \Rightarrow |x| > 2 \Rightarrow x < -2$ or $x > 2$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n = \frac{1}{1-(2/x)} = \frac{x}{x-2}, x > 2 \text{ or } x < -2$$

**63.** $\sum_{n=1}^{\infty}(x-1)^n = (x-1)\sum_{n=0}^{\infty}(x-1)^n$

Geometric series: converges for $|x-1| < 1 \Rightarrow 0 < x < 2$

$$f(x) = (x-1)\sum_{n=0}^{\infty}(x-1)^n$$
$$= (x-1)\frac{1}{1-(x-1)} = \frac{x-1}{2-x}, \quad 0 < x < 2$$

**64.** $\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$

Geometric series: converges for

$\left|\frac{x-2}{3}\right| < 1 \Rightarrow |x-2| < 3 \Rightarrow -1 < x < 5$

$$f(x) = \sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$$
$$= \frac{5}{1-\left(\frac{x-2}{3}\right)} = \frac{5}{(3-x+2)/3}$$
$$= \frac{15}{5-x}, -1 < x < 5$$

**65.** $\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n$

Geometric series: converges for

$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

$$f(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1+x}, \quad -1 < x < 1$$

**66.** $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}\left(-x^2\right)^n$

Geometric series: converges for

$\left|-x^2\right| < 1 \Rightarrow -1 < x < 1$

$$f(x) = \sum_{n=0}^{\infty}\left(-x^2\right)^n = \frac{1}{1-\left(-x^2\right)} = \frac{1}{1+x^2}, -1 < x < 1$$

**67.** (a) $x$ is the common ratio.

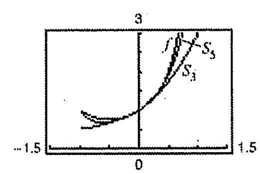
(b) $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$

(c) $y_1 = \frac{1}{1-x}$

$y_2 = S_3 = 1 + x + x^2$

$y_3 = S_5 = 1 + x + x^2 + x^3 + x^4$

Answers will vary.

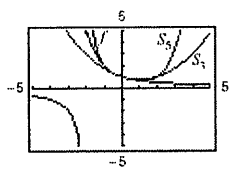


**68.** (a) $\left(-\frac{x}{2}\right)$ is the common ratio.

(b) $$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$
$$= \frac{1}{1-(-x/2)}$$
$$= \frac{2}{2+x}, \quad |x| < 2$$

(c) $$y_1 = \frac{2}{2+x}$$
$$y_2 = S_3 = 1 - \frac{x}{2} + \frac{x^2}{4}$$
$$y_3 = S_5 = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}$$

Answers will vary.




**69.** $$\frac{1}{n(n+1)} < 0.0001$$
$$10{,}000 < n^2 + n$$
$$0 < n^2 + n - 10{,}000$$
$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10{,}000)}}{2}$$

Choosing the positive value for $n$ you have $n \approx 99.5012$. The first *term* that is less than 0.0001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.0001$$
$$10{,}000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

**70.** $$\frac{1}{2^n} < 0.0001$$
$$10{,}000 < 2^n$$

This inequality is true when $n = 14$.

$$(0.01)^n < 0.0001$$
$$10{,}000 < 10^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

**71.** $$\sum_{i=0}^{n-1} 8000(0.95)^i = \frac{8000\left[1 - 0.95^n\right]}{1 - 0.95}$$
$$= 160{,}000\left[1 - 0.95^n\right], \quad n > 0$$

**72.** $$V(t) = 475{,}000(1 - 0.3)^n = 475{,}000(0.7)^n$$
$$V(5) = 475{,}000(0.7)^5 = \$79{,}833.25$$

**73.** $$\sum_{i=0}^{\infty} 200(0.75)^i = 800 \text{ million dollars}$$

**74.** $$\sum_{i=0}^{\infty} 200(0.60)^i = 500 \text{ million dollars}$$

**75.** $$D_1 = 16$$
$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$
$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$
$$\vdots$$
$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots$$
$$= -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81}$$
$$\approx 152.42 \text{ feet}$$

**76.** The ball in Exercise 75 takes the following times for each fall.

| | |
|---|---|
| $s_1 = -16t^2 + 16$ | $s_1 = 0$ if $t = 1$ |
| $s_2 = -16t^2 + 16(0.81)$ | $s_2 = 0$ if $t = 0.9$ |
| $s_3 = -16t^2 + 16(0.81)^2$ | $s_3 = 0$ if $t = (0.9)^2$ |
| $\vdots$ | $\vdots$ |
| $s_n = -16t^2 + 16(0.81)^{n-1}$ | $s_n = 0$ if $t = (0.9)^{n-1}$ |

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n = -1 + 2\sum_{n=0}^{\infty}(0.9)^n$$
$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

**77.** $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$

**78.** $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

$P(2) = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$

$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

**79.** (a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1}{2}\frac{1}{(1 - (1/2))} = 1$

(b) No, the series is not geometric.

(c) $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$

**80.** Person 1: $\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{2}\frac{1}{1 - (1/8)} = \frac{4}{7}$

Person 2: $\frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{4}\frac{1}{1 - (1/8)} = \frac{2}{7}$

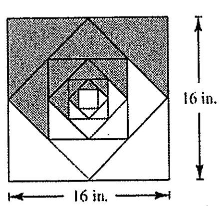
Person 3: $\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{8}\frac{1}{1 - (1/8)} = \frac{1}{7}$

Sum: $\frac{4}{7} + \frac{2}{7} + \frac{1}{7} = 1$

**81.** (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$

**Note:** This is one-half of the area of the original square




**82.** (a) $\sin\theta = \frac{|Y_{y_1}|}{z} \Rightarrow |Y_{y_1}| = z\sin\theta$

$\sin\theta = \frac{|x_1y_1|}{|Y_{y_1}|} \Rightarrow |x_1y_1| = |Y_{y_1}|\sin\theta = z\sin^2\theta$

$\sin\theta = \frac{|x_1y_2|}{|x_1y_1|} \Rightarrow |x_1y_2| = |x_1y_1|\sin\theta = z\sin^3\theta$

Total: $z\sin\theta + z\sin^2\theta + z\sin^3\theta + \cdots = z\frac{\sin\theta}{1 - \sin\theta}$

(b) If $z = 1$ and $\theta = \frac{\pi}{6}$, then total $= \frac{1/2}{1 - (1/2)} = 1$.

**83.** $\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n = \frac{100{,}000}{1.06}\sum_{i=0}^{19}\left(\frac{1}{1.06}\right)^i = \frac{100{,}000}{1.06}\left[\frac{1-1.06^{-20}}{1-1.06^{-1}}\right] \quad \left(n = 20, r = 1.06^{-1}\right) \approx \$1{,}146{,}992.12$

The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

**84.**
$$\sum_{n=0}^{12t-1} P\left(1+\frac{r}{12}\right)^n = \frac{P\left[1-\left(1+\frac{r}{12}\right)^{12t}\right]}{1-\left(1+\frac{r}{12}\right)}$$
$$= P\left(-\frac{12}{r}\right)\left[1-\left(1+\frac{r}{12}\right)^{12t}\right]$$
$$= P\left(\frac{12}{r}\right)\left[\left(1+\frac{r}{12}\right)^{12t}-1\right]$$
$$\sum_{n=0}^{12t-1} P\left(e^{r/12}\right)^n = \frac{P\left(1-\left(e^{r/12}\right)^{12t}\right)}{1-e^{r/12}} = \frac{P\left(e^{rt}-1\right)}{e^{r/12}-1}$$

**85.** $w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01\left(1-2^n\right)}{1-2} = 0.01\left(2^n-1\right)$

(a) When $n = 29$: $w = \$5{,}368{,}709.11$

(b) When $n = 30$: $w = \$10{,}737{,}418.23$

(c) When $n = 31$: $w = \$21{,}474{,}836.47$

**86.** Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4(\pi + \pi + \cdots) = \infty$

**87.** $P = 45, \quad r = 0.03, \quad t = 20$

(a) $A = 45\left(\frac{12}{0.03}\right)\left[\left(1+\frac{0.03}{12}\right)^{12(20)}-1\right] \approx \$14{,}773.59$

(b) $A = \frac{45\left(e^{0.03(20)}-1\right)}{e^{0.03/12}-1} \approx \$14{,}779.65$

**88.** $P = 75, \quad r = 0.055, \quad t = 25$

(a) $A = 75\left(\frac{12}{0.055}\right)\left[\left(1+\frac{0.055}{12}\right)^{12(25)}-1\right] \approx \$48{,}152.81$

(b) $A = \frac{75\left(e^{0.055(25)}-1\right)}{e^{0.055/12}-1} \approx \$48{,}245.07$

**89.** $P = 100, \quad r = 0.04, \quad t = 35$

(a) $A = 100\left(\frac{12}{0.04}\right)\left[\left(1+\frac{0.04}{12}\right)^{12(35)}-1\right] \approx \$91{,}373.09$

(b) $A = \frac{100\left(e^{0.04(35)}-1\right)}{e^{0.04/12}-1} \approx \$91{,}503.32$

**90.** $P = 30, \quad r = 0.06, \quad t = 50$

(a) $A = 30\left(\frac{12}{0.06}\right)\left[\left(1+\frac{0.06}{12}\right)^{12(50)}-1\right] \approx 113{,}615.73$

(b) $A = \frac{30\left(e^{0.06(50)}-1\right)}{e^{0.06/12}-1} \approx \$114{,}227.18$

**91.** False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**92.** True

**93.** False; $\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r}\right) - a$

The formula requires that the geometric series begins with $n = 0$.

**94.** True

$$\lim_{n\to\infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$$

**95.** True

$$0.74999\ldots = 0.74 + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$$
$$= 0.74 + \frac{9}{10^3}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$$
$$= 0.74 + \frac{9}{10^3}\cdot\frac{1}{1-(1/10)}$$
$$= 0.74 + \frac{9}{10^3}\cdot\frac{10}{9}$$
$$= 0.74 + \frac{1}{100} = 0.75$$

**96.** True

**97.** Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum(a_n + b_n) = \sum_{n=0}^{\infty}[1 + (-1)] = \sum_{n=0}^{\infty}[1 - 1] = 0$$

**98.** If $\Sigma(a_n + b_n)$ converged, then $\Sigma(a_n + b_n) - \Sigma a_n = \Sigma b_n$ would converge, which is a contradiction. So, $\Sigma(a_n + b_n)$ diverges.

**99.** (a) $\dfrac{1}{a_{n+1}a_{n+2}} - \dfrac{1}{a_{n+2}a_{n+3}} = \dfrac{a_{n+3} - a_{n+1}}{a_{n+1}a_{n+2}a_{n+3}} = \dfrac{a_{n+2}}{a_{n+1}a_{n+2}a_{n+3}} = \dfrac{1}{a_{n+1}a_{n+3}}$

(b) 
$$S_n = \sum_{k=0}^{n} \frac{1}{a_{k+1}a_{k+3}}$$
$$= \sum_{k=0}^{n}\left[\frac{1}{a_{k+1}a_{k+2}} - \frac{1}{a_{k+2}a_{k+3}}\right]$$
$$= \left[\frac{1}{a_1a_2} - \frac{1}{a_2a_3}\right] + \left[\frac{1}{a_2a_3} - \frac{1}{a_3a_4}\right] + \cdots + \left[\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}}\right] = \frac{1}{a_1a_2} - \frac{1}{a_{n+2}a_{n+3}} = 1 - \frac{1}{a_{n+2}a_{n+3}}$$

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \frac{1}{a_{n+2}a_{n+3}}\right] = 1$$

**100.** Let $\{S_n\}$ be the sequence of partial sums for the convergent series

$\sum_{n=1}^{\infty} a_n = L$. Then $\lim_{n\to\infty} S_n = L$ and because

$$R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

you have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(L - S_n) = \lim_{n\to\infty} L - \lim_{n\to\infty} S_n = L - L = 0.$$

**101.** $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \dfrac{1}{r}\left(\dfrac{1}{r}\right)^n = \dfrac{1/r}{1 - (1/r)} = \dfrac{1}{r - 1} \quad \left(\text{since}\left|\dfrac{1}{r}\right| < 1\right)$

This is a geometric series which converges if

$\left|\dfrac{1}{r}\right| < 1 \Leftrightarrow |r| > 1.$

**102.** The entire rectangle has area 2 because the height is 1 and the base is $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$. The squares all lie inside the rectangle, and the sum of their areas is

$1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots.$

So, $\sum_{n=1}^{\infty} \dfrac{1}{n^2} < 2.$

**103.** The series is telescoping:

$$S_n = \sum_{k=1}^{n} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$
$$= \sum_{k=1}^{n}\left[\frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}}\right]$$
$$= 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}}$$
$$\lim_{n\to\infty} S_n = 3 - 1 = 2$$