9.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Partial Sums In Exercises 1–6, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$
2. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$
3. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$
4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$
5. $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Verifying Divergence In Exercises 7–14, verify that the infinite series diverges.

7. $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$
8. $\sum_{n=0}^{\infty} 4(-1.05)^n$
9. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
10. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
11. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$
12. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$
13. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$
14. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

Verifying Convergence In Exercises 15–20, verify that the infinite series converges.

15. $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$
16. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$
17. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$
18. $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$
19. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (*Hint*: Use partial fractions.)
20. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (*Hint*: Use partial fractions.)

Numerical, Graphical, and Analytic Analysis In Exercises 21–24, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

21. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$
22. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$
23. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$
24. $\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1}$

Finding the Sum of a Convergent Series In Exercises 25–34, find the sum of the convergent series.

25. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$
26. $\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$
27. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$
28. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$
29. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$
30. $9 - 3 + 1 - \frac{1}{3} + \cdots$
31. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$
32. $\sum_{n=0}^{\infty} [(0.3)^n + (0.8)^n]$
33. $\sum_{n=1}^{\infty} (\sin 1)^n$
34. $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 3n - 2}$

Using a Geometric Series In Exercises 35–40, (a) write the repeating decimal as a geometric series, and (b) write its sum as the ratio of two integers.

35. $0.\overline{4}$
36. $0.\overline{36}$
37. $0.\overline{81}$
38. $0.\overline{01}$
39. $0.0\overline{75}$
40. $0.2\overline{15}$

Determining Convergence or Divergence In Exercises 41–54, determine the convergence or divergence of the series.

41. $\sum_{n=0}^{\infty} (1.075)^n$
42. $\sum_{n=0}^{\infty} \frac{3^n}{1000}$
43. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$
44. $\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$
45. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$
46. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$
47. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$
48. $\sum_{n=0}^{\infty} \frac{3}{5^n}$
49. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$
50. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$
51. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$
52. $\sum_{n=1}^{\infty} e^{-n}$
53. $\sum_{n=1}^{\infty} \arctan n$
54. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

WRITING ABOUT CONCEPTS

55. **Series** State the definitions of convergent and divergent series.
56. **Sequence and Series** Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.

WRITING ABOUT CONCEPTS (continued)

57. Geometric Series Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

58. *n*th-Term Test for Divergence State the nth-Term Test for Divergence.

59. Comparing Series Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

60. Using a Series

(a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

Making a Series Converge **In Exercises 61–66, find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.**

61. $\sum_{n=1}^{\infty} (3x)^n$

62. $\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n$

63. $\sum_{n=1}^{\infty} (x-1)^n$

64. $\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$

65. $\sum_{n=0}^{\infty} (-1)^n x^n$

66. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

Using a Geometric Series **In Exercises 67 and 68, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums S_3 and S_5. What do you notice?**

67. $1 + x + x^2 + x^3 + \cdots$

68. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

Writing **In Exercises 69 and 70, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

69. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$

70. $\sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} (0.01)^n$

71. Marketing An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year, 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.95(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after n years?

72. Depreciation A company buys a machine for \$475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after n years. What is its value after 5 years?

73. Multiplier Effect

The total annual spending by tourists in a resort city is \$200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$200 million and find the sum of the series.

74. Multiplier Effect Repeat Exercise 73 when the percent of the revenue that is spent again in the city decreases to 60%.

75. Distance A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

76. Time The ball in Exercise 75 takes the following times for each fall.

$s_1 = -16t^2 + 16,\qquad s_1 = 0 \text{ when } t = 1$

$s_2 = -16t^2 + 16(0.81),\qquad s_2 = 0 \text{ when } t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2,\qquad s_3 = 0 \text{ when } t = (0.9)^2$

$s_4 = -16t^2 + 16(0.81)^3,\qquad s_4 = 0 \text{ when } t = (0.9)^3$

$\vdots \qquad\qquad \vdots$

$s_n = -16t^2 + 16(0.81)^{n-1},\qquad s_n = 0 \text{ when } t = (0.9)^{n-1}$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$$

Find this total time.

Probability **In Exercises 77 and 78, the random variable n represents the number of units of a product sold per day in a store. The probability distribution of n is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(0) + P(1) + P(2) + P(3) + \cdots = 1$.**

77. $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

78. $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

79. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

(b) The expected number of tosses required until the first head occurs in the experiment is given by

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$$

Is this series geometric?

(c) Use a computer algebra system to find the sum in part (b).

AISPIX by Image Source/Shutterstock.com

80. Probability In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

81. Area The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) when this process is continued five more times, and (b) when this pattern of shading is continued infinitely.

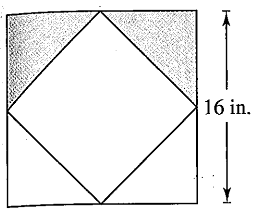


Figure for 81

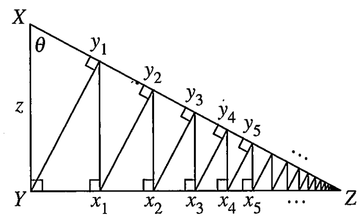


Figure for 82

82. Length A right triangle XYZ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

(b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \pi/6$.

Using a Geometric Series **In Exercises 83–86, use the formula for the *n*th partial sum of a geometric series**

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}.$$

83. Present Value The winner of a \$2,000,000 sweepstakes will be paid \$100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is $\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n$. Compute the present value and interpret its meaning.

84. Annuities When an employee receives a paycheck at the end of each month, P dollars is invested in a retirement account. These deposits are made each month for t years and the account earns interest at the annual percentage rate r. When the interest is compounded monthly, the amount A in the account at the end of t years is

$$A = P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{12t-1}$$
$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right].$$

When the interest is compounded continuously, the amount A in the account after t years is

$$A = P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12}$$
$$= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}.$$

Verify the formulas for the sums given above.

85. Salary You go to work at a company that pays \$0.01 for the first day, \$0.02 for the second day, \$0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

86. Sphereflake

The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.

Annuities **In Exercises 87–90, consider making monthly deposits of P dollars in a savings account at an annual interest rate r. Use the results of Exercise 84 to find the balance A after t years when the interest is compounded (a) monthly and (b) continuously.**

87. $P = \$45,\quad r = 3\%,\quad t = 20$ years

88. $P = \$75,\quad r = 5.5\%,\quad t = 25$ years

89. $P = \$100,\quad r = 4\%,\quad t = 35$ years

90. $P = \$30,\quad r = 6\%,\quad t = 50$ years

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

92. If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

93. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \frac{a}{1 - r}$.

94. The series $\sum_{n=1}^{\infty} \frac{n}{1000(n + 1)}$ diverges.

95. $0.75 = 0.749999\ldots$

96. Every decimal with a repeating pattern of digits is a rational number.

Courtesy of Eric Haines

97. Using Divergent Series Find two divergent series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma(a_n + b_n)$ converges.

98. Proof Given two infinite series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n$ converges and $\Sigma\, b_n$ diverges, prove that $\Sigma(a_n + b_n)$ diverges.

99. Fibonacci Sequence The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\dfrac{1}{a_{n+1}\,a_{n+3}} = \dfrac{1}{a_{n+1}\,a_{n+2}} - \dfrac{1}{a_{n+2}\,a_{n+3}}$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a_{n+1}\,a_{n+3}} = 1$.

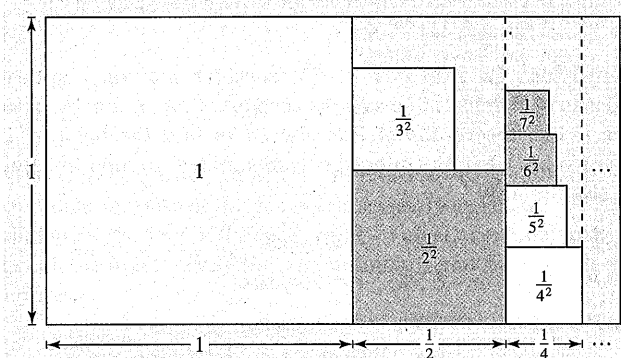
100. Remainder Let $\Sigma\, a_n$ be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdot \cdot \cdot$$

be the remainder of the series after the first N terms. Prove that $\displaystyle\lim_{N\to\infty} R_N = 0$.

101. Proof Prove that $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdot \cdot \cdot = \dfrac{1}{r-1}$, for $|r| > 1$.

102. HOW DO YOU SEE IT? The figure below represents an informal way of showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$. Explain how the figure implies this conclusion.



FOR FURTHER INFORMATION For more on this exercise, see the article "Convergence with Pictures" by P. J. Rippon in *American Mathematical Monthly.*

PUTNAM EXAM CHALLENGE

103. Express $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number.

104. Let $f(n)$ be the sum of the first n terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, . . . , where the nth term is given by

$$a_n = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

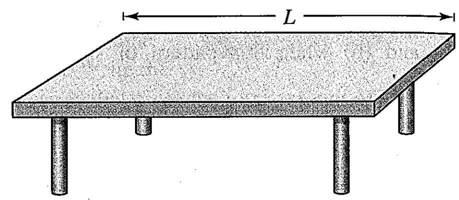
Show that if x and y are positive integers and $x > y$ then $xy = f(x + y) - f(x - y)$.

SECTION PROJECT

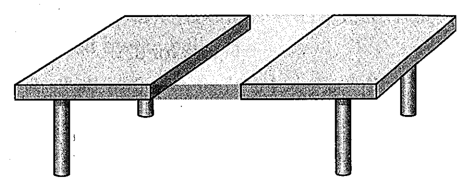
Cantor's Disappearing Table

The following procedure shows how to make a table disappear by removing only half of the table!

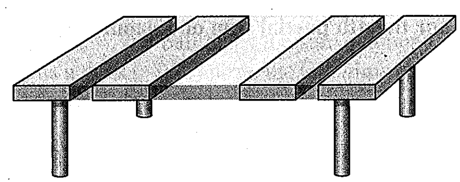
(a) Original table has a length of L.



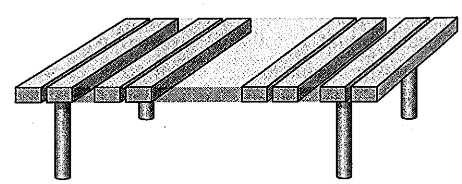
(b) Remove $\frac{1}{4}$ of the table centered at the midpoint. Each remaining piece has a length that is less than $\frac{1}{2}L$.



(c) Remove $\frac{1}{8}$ of the table by taking sections of length $\frac{1}{16}L$ from the centers of each of the two remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8}$ of the table. Each remaining piece has a length that is less than $\frac{1}{4}L$.



(d) Remove $\frac{1}{16}$ of the table by taking sections of length $\frac{1}{64}L$ from the centers of each of the four remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ of the table. Each remaining piece has a length that is less than $\frac{1}{8}L$.



Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

FOR FURTHER INFORMATION Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*