Key

BC Calculus – 9.3 Notes – Finding Arc Lengths (Parametric Equations)

Recall: Arc Length

$$1+\left(\frac{dy/dt}{dx/dt}\right)^2$$

$$\left(\left(\frac{\frac{dx}{dt}}{dx/dt}\right)^2+\left(\frac{\frac{dy}{dt}}{dx/dt}\right)^2\right)$$

$$\left(\frac{dt}{dx}\right)^2\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]$$

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dt}\right)^2}\,dx$$

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot\frac{dt}{\cancel{dx}}\cdot\cancel{dx}$$

Arc Length in Parametric Form (for smooth curves)

$$\boxed{L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

For each set of parametric equations, find the length of the curve on the given interval.

1. $x(t)=\cos t$ and $y(t)=\sin t$ on the interval $0\le t\le 2\pi$.

$x'(t)=-\sin(t)\qquad y'(t)=\cos(t)$

$$\int_0^{2\pi}\sqrt{(\sin t)^2+(\cos t)^2}\,dt$$

$$\int\sqrt{1}\,dt$$

$$\int 1\,dt\to t\Big]_0^{2\pi}=2\pi-0$$

$\boxed{\text{Arc Length is } 2\pi}$

circumference of a circle of radius $r=1$

$(x^2+y^2=1)$

$C=2\pi r\to C=2\pi(1)=2\pi$

2. $x=1-4t$ and $y=7t$ on the interval $0\le t\le 2$.

$x'(t)=-4\qquad y'(t)=7$

$$\int_0^2\sqrt{(-4)^2+(7)^2}\,dt$$

$$\int_0^2\sqrt{65}\,dt$$

$$\sqrt{65}\,t\Big]_0^2=2\sqrt{65}-0$$

$\boxed{\text{Arc Length}=2\sqrt{65}}$

9.3 Arc Length (Parametric Form)

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Calculus

Practice

What is the length of the curve defined by the parametric equations? Solve without the use of a calculator.

1. $x(t) = 6t + 10$ and $y(t) = 14 - 4t$ for the interval $-1 \le t \le 3$?

$$L=\int_{-1}^{3}\sqrt{(6)^2+(-4)^2}\,dt = \int_{-1}^{3}\sqrt{52}\,dt$$

$$\sqrt{52}t\Big]_{-1}^{3} = 3\sqrt{52}-(-1\sqrt{52})$$

$$=\boxed{4\sqrt{52} \text{ or } 4\cdot\sqrt{4\cdot 13} \text{ or } 8\sqrt{13}}$$

2. $x = \frac{a}{2}t^2$ and $y = \frac{b}{2}t^2$, where a and b are constants. What is the length of the curve from $t = 0$ to $t = 1$?

$x'(t)=2\cdot\frac{a}{2}t$

$y'(t)=2\cdot\frac{b}{2}t$

$$\int_0^1\sqrt{(at)^2+(bt)^2}\,dt$$

$$\int\sqrt{t^2(a^2+b^2)}\,dt$$

$$\int_0^1 t\sqrt{a^2+b^2}\,dt \rightarrow \frac{t^2}{2}\left(\sqrt{a^2+b^2}\right)\Big]_0^1$$

$$\boxed{\frac{1}{2}\sqrt{a^2+b^2}-0}$$

3. $x(t) = 2t^2$ and $y(t) = \frac{2}{3}t^3$ for the interval $1 \le t \le 4$?

$$L=\int_1^4\sqrt{(4t)^2+(2t^2)^2}\,dt$$

$$\int\sqrt{16t^2+4t^4}\,dt \rightarrow \sqrt{4t^2(4+t^2)}\,dt$$

$$\int 2t\sqrt{4+t^2}\,dt \qquad u=4+t^2 \quad \frac{du}{dt}=2t \quad \Big|\; dt=\frac{du}{2t}$$

$$\int \cancel{2t}\,u^{1/2}\cdot\frac{du}{\cancel{2t}}$$

$$\frac{u^{3/2}}{3/2} \rightarrow \frac{2}{3}(4+t^2)^{3/2}\Big]_1^4 = \boxed{\frac{2}{3}\left[20^{3/2}-5^{3/2}\right] = \frac{70}{3}\sqrt{5}}$$

4. $x(\theta) = 5\cos\theta$ and $y(\theta) = 5\sin\theta$ for the interval $0 \le \theta \le 2\pi$.

$$\int_0^{2\pi}\sqrt{(-5\sin\theta)^2+(5\cos\theta)^2}\,d\theta$$

$$\int\sqrt{25(\sin^2\theta+\cos^2\theta)}\,d\theta = \int_0^{2\pi}\sqrt{25}\,d\theta$$

$$\int 5\,d\theta \rightarrow 5\theta\Big]_0^{2\pi} = 10\pi - 0$$

$$=\boxed{10\pi}$$

5. $x(t) = 7t - 2$ and $y(t) = 4 - 8t$ for the interval $1 \le t \le 5$.

$$\int_1^5\sqrt{(7)^2+(-8)^2}\,dt$$

$$\int_1^5\sqrt{113}\,dt$$

$$\sqrt{113}t\Big]_1^5 = 5\sqrt{113}-1\sqrt{113}$$

$$=\boxed{4\sqrt{113}}$$

6. If a curved is described by the parametric equations $x = t^2$ and $y = 2e^{2t}$, then which of the following gives the length of the path from $t = 0$ to $t = \ln 3$?

$$\int \sqrt{(2t)^2+(4e^{2t})^2}\,dt$$

A. $\int_0^{\ln 3} \sqrt{4t^2 + 4e^{4t}}\,dt$

B. $\int_0^{\ln 3} \sqrt{t^4 + 4e^{4t}}\,dt$

C. $\int_0^{\ln 3} \sqrt{4t^2 + 16e^{4t}}\,dt$

D. $\int_0^{\ln 3} \sqrt{t^2 + 2e^{2t}}\,dt$

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7. Which of the following gives the length of the path described by the parametric equations $x = 2 + 4t$ and $y = 3 + t^2$ from $t = 0$ to $t = 1$?

$$\int \sqrt{(4)^2+(2t)^2}\,dt$$

A. $\int_0^1 \sqrt{4 + 2t}\,dt$

B. $\int_0^1 \sqrt{(2 + 4t)^2 + (3 + t^2)^2}\,dt$

C. $\int_0^1 \sqrt{16t^2 + t^4}\,dt$

D. $\int_0^1 \sqrt{16 + 4t^2}\,dt$

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8. Which of the following gives the length of the path described by the parametric equations $x = \cos t^3$ and $y = e^{5t}$ from $t = 0$ to $t = \pi$?

$$\int \sqrt{[-\sin(t^3)3t^2]^2+(5e^{5t})^2}\,dt$$

A. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{10t}}\,dt$

B. $\int_0^{\pi} \sqrt{-3t^2 \sin(t^3) + 5e^{5t}}\,dt$

C. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{5t}}\,d$

D. $\int_0^{\pi} \sqrt{(\cos(t^3))^2 + (e^{5t})^2}\,dt$

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9. Which of the following gives the length of the path described by the parametric equations $x = \sin 3t$ and $y = \cos 2t$ from $t = 0$ to $t = \pi$?

$$L = \sqrt{[3\cos(3t)]^2+[-2\sin(2t)]^2}\,dt$$

A. $\int_0^{\pi} \sqrt{\sin^2 3t + \cos^2 2t}\,dt$

B. $\int_0^{\pi} \sqrt{\cos^2 3t + \sin^2 2t}\,dt$

C. $\int_0^{\pi} \sqrt{9\cos 3t + 4\sin 2t}\,dt$

D. $\int_0^{\pi} \sqrt{9\cos^2 3t + 4\sin^2 2t}\,dt$

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10. Which of the following gives the length of the path described by the parametric equations $x = \sqrt{t}$ and $y = 3t - 1$ from $0 \le t \le 1$?

$$L = \int \sqrt{[\tfrac{1}{2}t^{-1/2}]^2+[3]^2}\,dt$$

A. $\int_0^1 \sqrt{\frac{t}{4} + 9}\,dt$

B. $\int_0^1 \sqrt{\frac{1}{4}t^{-1} + 9}\,dt$

C. $\int_0^1 \sqrt{\frac{1}{4}t + 3}\,dt$

D. $\int_0^1 \sqrt{\frac{1}{2}t^{-\frac{1}{2}} + 3}\,dt$

No test prep. Problems 6-10 are great examples of problems you may see on the AP Exam.