# AP Calculus BC 9.3 Notes Integral Test and p-Series Test

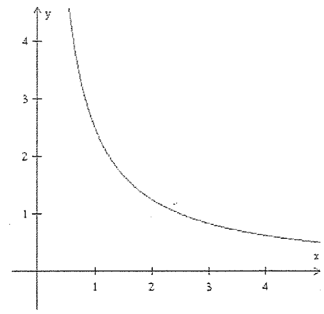
*Use the Integral Test to determine whether an infinite series converges or diverges.

**The Integral Test**

If $f$ is **positive**, **continuous**, and **decreasing** for $x \ge 1$

and $a_n = f(x)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$

either both converge or both diverge.



Note 1: This does NOT mean that the series converges to the value of the definite integral!!!!!!

Note 2: The function need only be decreasing for all $x > k$ for some $k \ge 1$

Determine whether the following series converge or diverge. If they converge, find an interval in which the sum resides using $S_4$ .

**Ex. 1:** Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

**Ex. 2:** Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

**Ex. 3:** Apply the Integral Test to the series $\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \frac{\ln 5}{\sqrt{5}} + \ldots$

## p-Series Test -Use properties of the p-series and harmonic series

**Definition of a p-Series**

A p-series is a type of series that follows the following pattern: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

where p is a positive constant. For p = 1, the series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$ is called the **harmonic series**.

**Convergence of p-Series**

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ a) **converges** if p > 1 b) **diverges** if 0 < p < 1, c) **diverges** if p = 1 (harmonic series)

Note: If the p-series converges, we cannot find the sum.

**Example 4:** Convergent and Divergent p-Series

(a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$ (d) $\sum_{n=1}^{\infty} \frac{999999999}{n^{1.0000000001}}$

e) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ g) $\sum_{n=1}^{\infty} \frac{1}{n}$

**Example 5: Testing a Series for Convergence**

Determine whether the following series converges or diverges

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Key

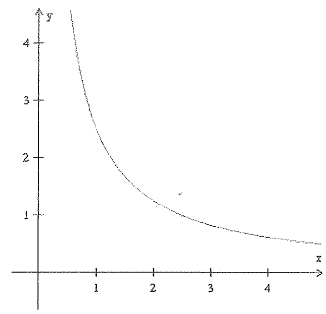
**AP Calculus BC** **9.3 Notes** **Integral Test and p-Series Test**

*Use the Integral Test to determine whether an infinite series converges or diverges.

**The Integral Test**

If $f$ is **positive, continuous,** and **decreasing** for $x \geq 1$ and $a_n = f(x)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

Improper Integral




Note 1: This does NOT mean that the series converges to the value of the definite integral!!!!!!

Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$ (long term behavior)

**Ex. 1:** Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

*$n^{th}$ term test, inconclusive
*not geometric, not telescoping

$\int_1^{\infty} \frac{x}{x^2+1}dx$ $u = x^2+1$ $dx = \frac{du}{2x}$ $\frac{du}{dx} = 2x$

$\int \frac{x}{u} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{u} du$

$\lim_{b \to \infty} \frac{1}{2}\ln(x^2+1)\Big]_1^b = \frac{1}{2}\ln(b^2) - \frac{1}{2}\ln(2)$

Since $\frac{n}{n^2+1}$ is positive, continuous, and decreasing, the sum diverges by the Integral Test.

$\lim_{b \to \infty} \frac{1}{2}\ln(b) - \frac{1}{2}\ln(2) = \boxed{\infty}$

**Ex. 2:** Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\lim_{b \to \infty} \int_1^b \frac{1}{x^2+1}$ $\lim_{b \to \infty} \arctan x\Big]_1^b$

$\lim_{b \to \infty} \arctan b - \arctan 1$

$= \frac{\pi}{2} - \frac{\pi}{4} \leftarrow$ finite.

So the sum converges by Integral Test since $\frac{1}{n^2+1}$ is positive, decreasing, continuous

**Ex. 3:** Apply the Integral Test to the series $\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \frac{\ln 5}{\sqrt{5}} + \ldots$

$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{\sqrt{n+1}}$ is positive, continuous, decreasing for $x > 0$ $\sum_{x=2}^{\infty} \frac{\ln x}{\sqrt{x}}$

$\lim_{b \to \infty} \int_1^b \frac{\ln(x+1)}{\sqrt{x+1}} =$

$\int_1^{\infty} \frac{\ln x}{\sqrt{x}} dx$ $x = e^u$ $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\int \frac{u}{\sqrt{x}} \cdot x\,du = \int u \cdot \sqrt{x}\,dx$

## The p-Series Test

$\int_1^\infty \frac{1}{x}\,dx$ diverges

### Definition of a p-Series

A p-series is a type of series that follows the following pattern: $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$

where p is a positive constant. For p = 1, the series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots$ is called the **harmonic series**.

(divergent series)

### Convergence of p-Series

p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ a) **converges** if p > 1 b) **diverges** if p < 1, c) **diverges** if p = 1 **(harmonic series)**

Note: If the p-series converges, we cannot find the sum.

**Example 4:** Convergent and Divergent p-Series

(a) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$

$=\frac{1}{n\cdot n^{1/2}}=\frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ $p=3/2>1$

By p-series, series converges.

(b) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n}}=\frac{n}{n^{1/2}}=\frac{1}{n^{-1/2}}$

$p=-\frac{1}{2}<1$

series diverges

(c) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n}$ $\frac{1}{n^{1/2}}$

$p=1/2<1$

Series diverges

(d) $\sum_{n=1}^{\infty}\frac{999999999}{n^{1.0000000001}}$

$9999\ldots\sum\frac{1}{n^{1.0001}}$

$p=1.0001>1$

series converges.

e) $\sum_{n=1}^{\infty}\frac{1}{n^2}$

$p=2>1$,

series converges.

f) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $\frac{1}{n^{1/2}}$

$p=1/2<1$,

series diverges.

g) $\sum_{n=1}^{\infty}\frac{1}{n}$

$p=1$,

harmonic series, so series diverges.

**Example 5: Testing a Series for Convergence**

Determine whether the following series converges or diverges

$\frac{1}{n\ln n}$ is positive, decreasing, continuous for $x\geq 2$.

$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

$u=\ln x$

$\frac{du}{dx}=\frac{1}{x}$

$dx=x\,du$

$\int\frac{1}{\cancel{x}\cdot u}\cdot \cancel{x}\,du$

$\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}\,dx$

$\int\frac{1}{u}\,du=\lim_{b\to\infty}\ln(\ln x)\Big]_2^b=\lim_{b\to\infty}\ln(\ln b)-\ln(\ln 2)=\infty$

By Integral Test, Sum Diverges.