104. $f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 4, \dots$

In general: $f(n) = \begin{cases} n^2/4, & n \text{ even} \\ (n^2 - 1)/4, & n \text{ odd.} \end{cases}$

(See below for a proof of this.)

$x + y$ and $x - y$ are either both odd or both even. If both even, then

$$f(x + y) - f(x - y) = \frac{(x + y)^2}{4} - \frac{(x - y)^2}{4} = xy.$$

If both odd,

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - 1}{4} - \frac{(x - y)^2 - 1}{4} = xy.$$

Proof by induction that the formula for $f(n)$ is correct. It is true for $n = 1$. Assume that the formula is valid for k. If k is even, then $f(k) = k^2/4$ and

$$f(k + 1) = f(k) + \frac{k}{2} = \frac{k^2}{4} + \frac{k}{2} = \frac{k^2 + 2k}{4} = \frac{(k + 1)^2 - 1}{4}.$$

The argument is similar if k is odd.

Section 9.3 The Integral Test and *p*-Series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 3}$

Let

$f(x) = \dfrac{1}{x + 3}, \quad f'(x) = -\dfrac{1}{(x + 3)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x + 3}\,dx = \left[\ln(x + 3)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n + 5}$

Let $f(x) = \dfrac{2}{3x + 5}$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x + 5}\,dx = \left[\frac{2}{3}\ln(3x + 5)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Let $f(x) = \dfrac{1}{2^x}, \quad f'(x) = -(\ln 2)2^{-x} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2^x}\,dx = \left[\frac{-1}{(\ln 2)\,2^x}\right]_1^{\infty} = \frac{1}{2\ln 2}$$

So, the series converges by Theorem 9.10.

4. $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$

Let $f(x) = \dfrac{1}{3^x}, \quad f'(x) = -(\ln 3)3^{-x} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{3^x}\,dx = \left[\frac{-1}{(\ln 3)\,3^x}\right]_1^{\infty} = \frac{1}{3\ln 3}$$

So, the series converges by Theorem 9.10.

5. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}, \quad f'(x) = -e^{-x} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_1^{\infty} = \frac{1}{e}$$

So, the series converges by Theorem 9.10.

6. $\displaystyle\sum_{n=1}^{\infty} ne^{-n/2}$

Let $f(x) = xe^{-x/2}, \quad f'(x) = \dfrac{2 - x}{2e^{x/2}} < 0$ for $x \geq 3$.

f is positive, continuous, and decreasing for $x \geq 3$

$$\int_3^{\infty} xe^{-x/2}\,dx = \left[-2(x + 2)e^{-x/2}\right]_3^{\infty} = 10e^{-3/2}$$

So, the series converges by Theorem 9.10.

7. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

Let

$f(x) = \frac{1}{x^2 + 1}$, $f'(x) = -\frac{2x}{(x^2 + 1)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2 + 1}\,dx = \left[\arctan x\right]_1^{\infty} = \frac{\pi}{4}$$

So, the series converges by Theorem 9.10.

8. $\sum_{n=1}^{\infty} \frac{1}{2n + 1}$

Let

$f(x) = \frac{1}{2x + 1}$, $f'(x) = -\frac{2}{(2x + 1)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x + 1}\,dx = \left[\ln \sqrt{2x + 1}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

9. $\sum_{n=1}^{\infty} \frac{\ln(n + 1)}{n + 1}$

Let $f(x) = \frac{\ln(x + 1)}{x + 1}$, $f'(x) = \frac{1 - \ln(x + 1)}{(x + 1)^2} < 0$ for $x \geq 2$.

f is positive, continuous, and decreasing for $x \geq 2$.

$$\int_1^{\infty} \frac{\ln(x + 1)}{x + 1}\,dx = \left[\frac{\left[\ln(x + 1)\right]^2}{2}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

10. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Let $f(x) = \frac{\ln x}{\sqrt{x}}$, $f'(x) = \frac{2 - \ln x}{2x^{3/2}}$.

f is positive, continuous, and decreasing for $x > e^2 \approx 7.4$.

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx = \left[2\sqrt{x}(\ln x - 2)\right]_2^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

11. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n} + 1\right)}$

Let $f(x) = \frac{1}{\sqrt{x}\left(\sqrt{x} + 1\right)}$,

$f'(x) = -\frac{1 + 2\sqrt{x}}{2x^{3/2}\left(\sqrt{x} + 1\right)^2} < 0$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{\sqrt{x}\left(\sqrt{x} + 1\right)}\,dx = \left[2\ln\left(\sqrt{x} + 1\right)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

12. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 3}$

Let $f(x) = \frac{x}{x^2 + 3}$, $f'(x) = \frac{3 - x^2}{\left(x^2 + 3\right)} < 0$ for $x \geq 2$.

f is positive, continuous, and decreasing for $x \geq 2$

$$\int_1^{\infty} \frac{x}{x^2 + 3}\,dx = \left[\ln\sqrt{x^2 + 3}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

13. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + 1}$

Let $f(x) = \frac{\arctan x}{x^2 + 1}$,

$f'(x) = \frac{1 - 2x\arctan x}{\left(x^2 + 1\right)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{\arctan x}{x^2 + 1}\,dx = \left[\frac{(\arctan x)^2}{2}\right]_1^{\infty} = \frac{3\pi^2}{32}$$

So, the series converges by Theorem 9.10.

14. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \frac{\ln x}{x^3}, f'(x) = \frac{1 - 3\ln x}{x^4}$.

f is positive, continuous, and decreasing for $x > 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{(2\ln x + 1)}{4x^4}\right]_2^{\infty}$$
$$= \frac{2\ln 2 + 1}{16}$$

So, the series converges by Theorem 9.10.

15. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Let $f(x) = \frac{\ln x}{x^2}, f'(x) = \frac{1 - 2\ln x}{x^3}$.

f is positive, continuous, and decreasing for $x > e^{1/2} \approx 1.6$.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \left[\frac{-(\ln x + 1)}{x}\right]_1^{\infty} = 1$$

So, the series converges by Theorem 9.10.

16. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \frac{1}{x\sqrt{\ln x}}, f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}}$.

f is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \left[2\sqrt{\ln x}\right]_2^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

17. $\sum_{n=1}^{\infty} \frac{1}{(2n + 3)^3}$

Let $f(x) = (2x + 3)^{-3}, f'(x) = \frac{-6}{(2x + 3)^4} < 0$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} (2x + 3)^{-3}\,dx = \left[\frac{-1}{4(2x + 3)^2}\right]_1^{\infty} = \frac{1}{100}$$

So, the series converges by Theorem 9.10.

18. $\sum_{n=1}^{\infty} \frac{n + 2}{n + 1}$

Let $f(x) = \frac{x + 2}{x + 1} = 1 + \frac{1}{x + 1}, f'(x) = \frac{-1}{(x + 1)^2} < 0$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{x + 2}{x + 1}\,dx = \left[x + \ln(x + 1)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

[**Note:** $\lim_{n\to\infty} \frac{n + 2}{n + 1} = 1 \neq 0$, so the series diverges.]

19. $\sum_{n=1}^{\infty} \frac{4n}{2n^2 + 1}$

Let $f(x) = \frac{4x}{2x^2 + 1}, f'(x) = \frac{-4(2x^2 - 1)}{(2x^2 + 1)^2} < 0$ for $x \geq 1$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{4x}{2x^2 + 1}\,dx = \left[\ln(2x^2 + 1)\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

20. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n + 2}}$

Let $f(x) = \frac{1}{\sqrt{x + 2}}, f'(x) = \frac{-1}{2(x + 2)^{3/2}} < 0$.

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{(x + 2)^{1/2}}\,dx = \left[2\sqrt{x + 2}\right]_1^{\infty} = \infty$$

So, the series diverges by Theorem 9.10.

21. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

Let $f(x) = \frac{x}{x^4 + 1}, f'(x) = \frac{1 - 3x^4}{(x^4 + 1)^2} < 0$ for $x > 1$.

f is positive, continuous, and decreasing for $x > 1$.

$$\int_1^{\infty} \frac{x}{x^4 + 1}\,dx = \left[\frac{1}{2}\arctan(x^2)\right]_1^{\infty} = \frac{\pi}{8}$$

So, the series converges by Theorem 9.10.

22. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+2n^2+1}=\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}$

Let $f(x)=\dfrac{x}{(x^2+1)^2}$, $f'(x)=\dfrac{-(3x^2-1)}{(x^2+1)^3}<0$ for $x\ge 1$.

f is positive, continuous, and decreasing for $x\ge 1$.

$$\int_1^{\infty}\frac{x}{(x^2+1)^2}\,dx=\left[\frac{-1}{2(x^2+1)}\right]_1^{\infty}=\frac{1}{4}$$

So, the series converges by Theorem 9.10.

23. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{k-1}}{n^k+c}$

Let

$f(x)=\dfrac{x^{k-1}}{x^k+c}$, $f'(x)=\dfrac{x^{k-2}\left[c(k-1)-x^k\right]}{(x^k+c)^2}<0$

for $x>\sqrt[k]{c(k-1)}$.

f is positive, continuous, and decreasing for $x>\sqrt[k]{c(k-1)}$.

$$\int_1^{\infty}\frac{x^{k-1}}{x^k+c}\,dx=\left[\frac{1}{k}\ln(x^k+c)\right]_1^{\infty}=\infty$$

So, the series diverges by Theorem 9.10.

24. $\displaystyle\sum_{n=1}^{\infty}n^k e^{-n}$

Let $f(x)=\dfrac{x^k}{e^x}$, $f'(x)=\dfrac{x^{k-1}(k-x)}{e^x}<0$ for $x>k$.

f is positive, continuous, and decreasing for $x>k$.

Use integration by parts.

$$\int_1^{\infty}x^k e^{-x}\,dx=\left[-x^k e^{-x}\right]_1^{\infty}+k\int_1^{\infty}x^{k-1}e^{-x}\,dx$$
$$=\frac{1}{e}+\frac{k}{e}+\frac{k(k-1)}{e}+\cdots+\frac{k!}{e}$$

So, the series converges by Theorem 9.10.

25. Let $f(x)=\dfrac{(-1)^x}{x}$, $f(n)=a_n$.

The function f is not positive for $x\ge 1$.

26. Let $f(x)=e^{-x}\cos x$, $f(n)=a_n$.

The function f is not positive for $x\ge 1$.

27. Let $f(x)=\dfrac{2+\sin x}{x}$, $f(n)=a_n$.

The function f is not decreasing for $x\ge 1$.

28. Let $f(x)=\left(\dfrac{\sin x}{x}\right)^2$, $f(n)=a_n$.

The function f is not decreasing for $x\ge 1$.

29. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$

Let $f(x)=\dfrac{1}{x^3}$.

f is positive, continuous, and decreasing for $x\ge 1$.

$$\int_1^{\infty}\frac{1}{x^3}\,dx=\left[-\frac{1}{2x^2}\right]_1^{\infty}=\frac{1}{2}$$

Converges by Theorem 9.10

30. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

Let $f(x)=\dfrac{1}{x^{1/2}}=\dfrac{1}{\sqrt{x}}$.

f is positive, continuous, and decreasing for $x\ge 1$.

$$\int_1^{\infty}\frac{1}{x^{1/2}}\,dx=\left[2x^{1/2}\right]_1^{\infty}=\infty$$

Diverges by Theorem 9.10

31. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}$

Let $f(x)=\dfrac{1}{x^{1/4}}$, $f'(x)=\dfrac{-1}{4x^{5/4}}<0$ for $x\ge 1$

f is positive, continuous, and decreasing for $x\ge 1$.

$$\int_1^{\infty}\frac{1}{x^{1/4}}\,dx=\left[\frac{4x^{3/4}}{3}\right]_1^{\infty}=\infty$$

Diverges by Theorem 9.10

32. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$

Let $f(x)=\dfrac{1}{x^5}$.

f is positive, continuous, and decreasing for $x\ge 1$.

$$\int_1^{\infty}\frac{1}{x^5}\,dx=\left[-\frac{1}{4x^4}\right]_1^{\infty}=\frac{1}{4}$$

Converges by Theorem 9.10

33. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/5}}$

Divergent p-series with $p=\dfrac{1}{5}<1$

34. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent p-series with $p = \frac{5}{3} > 1$

35. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p-series with $p = \frac{3}{2} > 1$

36. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent p-series with $p = \frac{2}{3} < 1$

37. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p-series with $p = 1.04 > 1$

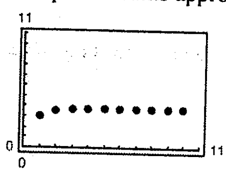
38. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent p-series with $p = \pi > 1$

39. (a)

n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75

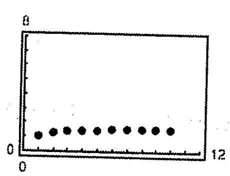
The partial sums approach the sum 3.75 very rapidly.



(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635

The partial sums approach the sum $\pi^2/6 \approx 1.6449$ slower than the series in part (a).



40. $\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Because the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

41. Let f be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge (Theorem 9.10). See Example 1, page 620.

42. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series, $p > 0$.

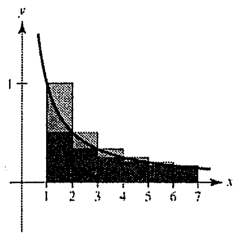
The p-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

43. Your friend is not correct. The series

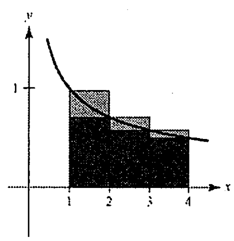
$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and therefore diverges.

44. $\sum_{n=1}^{6} a_n \geq \int_1^7 f(x)\,dx \geq \sum_{n=2}^{7} a_n$



45. (a)



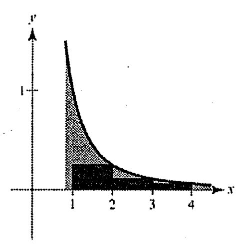
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$$

The area under the rectangle is greater than the area under the curve.

Because $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{\infty} = \infty$, diverges,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

(b)



$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$$

The area under the rectangles is less than the area under the curve.

Because $\int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$, converges,

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\left(\text{and so does } \sum_{n=1}^{\infty} \frac{1}{n^2}\right)$.

46. Answers will vary. *Sample answer*: The graph of the partial sums of the first series seems to be increasing without bound; therefore, the series diverges. The graph of the partial sums of the second series seems to be approaching a limit; therefore the series converges.

47. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$

48. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty}\frac{\ln x}{x^p}\,dx = \int_2^{\infty} x^{-p}\ln x\,dx = \left[\frac{x^{-p+1}}{(-p+1)^2}\left[-1 + (-p+1)\ln x\right]\right]_2^{\infty}. \text{ (Use integration by parts.)}$$

Converges for $-p + 1 < 0$ or $p > 1$

49. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(1+n^2)^p}$

If $p = 1$, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{1+n^2}$ diverges (see Example 1). Let

$$f(x) = \frac{x}{(1+x^2)^p},\ p \neq 1$$

$$f'(x) = \frac{1-(2p-1)x^2}{(1+x^2)^{p+1}}.$$

For a fixed $p > 0$, $p \neq 1$, $f'(x)$ is eventually negative. f is positive, continuous, and eventually decreasing.

$$\int_1^{\infty}\frac{x}{(1+x^2)^p}\,dx = \left[\frac{1}{(x^2+1)^{p-1}(2-2p)}\right]_1^{\infty}$$

For $p > 1$, this integral converges. For $0 < p < 1$, it diverges.

50. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

Because $p > 0$, the series diverges for all values of p.

51. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{p}\right)^n$, Geometric series.

Converges for $\left|\frac{3}{p}\right| < 1 \Rightarrow |p| > 3 \Rightarrow p > 3$

52. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n\ln n\left[\ln(\ln n)\right]^p}$

If $p = 1$, then

$$\int_3^{\infty}\frac{1}{x\ln x\left[\ln(\ln x)\right]}\,dx = \left[\ln(\ln(\ln x))\right]_3^{\infty} = \infty, \text{ so the}$$

series diverges by the Integral Test.

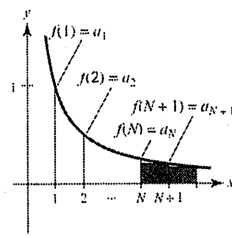
If $p \neq 1$,

$$\int_3^{\infty}\frac{1}{x\ln x\left[\ln(\ln x)\right]^p}\,dx = \left[\frac{\left[\ln(\ln x)\right]^{-p+1}}{-p+1}\right]_3^{\infty}.$$

This converges for $-p + 1 < 0 \Rightarrow p > 1$.

So, the series converges for $p > 1$, and diverges for $0 < p \le 1$.

53.



$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n > 0$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \cdots$$

$$\le \int_N^{\infty} f(x)\,dx$$

So, $0 \le R_n \le \int_N^{\infty} f(x)\,dx$

54. From Exercise 53, you have:

$$0 \le S - S_N \le \int_N^{\infty} f(x)\,dx$$

$$S_N \le S \le S_N + \int_N^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{N} a_n \le S \le \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx$$

55. $S_5 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \approx 1.4636$

$$0 \le R_5 \le \int_5^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_5^{\infty} = \frac{1}{5} = 0.2$$

$$1.4636 \le \sum_{n=1}^{\infty}\frac{1}{n^2} \le 1.4636 + 0.2 = 1.6636$$

56. $S_6 = 1 + \frac{1}{2^5} + \cdots + \frac{1}{6^5} \approx 1.0368$

$$0 \le R_6 \le \int_6^{\infty}\frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_6^{\infty} \approx 0.0002$$

$$1.0368 \le \sum_{n=1}^{\infty}\frac{1}{n^5} \le 1.0368 + 0.0002 = 1.0370$$

57. $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$0 \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$

$0.9818 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le 0.9818 + 0.0997 = 1.0815$

58. $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$0 \le R_{10} \le \int_{10}^{\infty} \frac{1}{(x+1)\left[\ln(x+1)\right]^3}\,dx = \left[-\frac{1}{2\left[\ln(x+1)\right]^2}\right]_{10}^{\infty} = \frac{1}{2(\ln 11)^2} \approx 0.0870$

$1.9821 \le \sum_{n=1}^{\infty} \frac{1}{(n+1)\left[\ln(n+1)\right]^3} \le 1.9821 + 0.0870 = 2.0691$

59. $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$0 \le R_4 \le \int_4^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^{\infty} = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$

$0.4049 \le \sum_{n=1}^{\infty} ne^{-n^2} \le 0.4049 + 5.6 \times 10^{-8}$

60. $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$0 \le R_4 \le \int_4^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_4^{\infty} \approx 0.0183$

$0.5713 \le \sum_{n=0}^{\infty} e^{-n} \le 0.5713 + 0.0183 = 0.5896$

61. $0 \le R_N \le \int_N^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_N^{\infty} = \frac{1}{3N^3} < 0.001$

$\frac{1}{N^3} < 0.003$

$N^3 > 333.33$

$N > 6.93$

$N \ge 7$

62. $0 \le R_N \le \int_N^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_N^{\infty} = \frac{2}{\sqrt{N}} < 0.001$

$N^{-1/2} < 0.0005$

$\sqrt{N} > 2000$

$N \ge 4{,}000{,}000$

63. $R_N \le \int_N^{\infty} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_N^{\infty} = \frac{2}{e^{N/2}} < 0.001$

$\frac{2}{e^{N/2}} < 0.001$

$e^{N/2} > 2000$

$\frac{N}{2} > \ln 2000$

$N > 2\ln 2000 \approx 15.2$

$N \ge 16$

64. $R_N \le \int_N^{\infty} \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_N^{\infty}$

$= \frac{\pi}{2} - \arctan N < 0.001$

$-\arctan N < 0.001 - \frac{\pi}{2}$

$\arctan N > \frac{\pi}{2} - 0.001$

$N > \tan\left(\frac{\pi}{2} - 0.001\right)$

$N \ge 1000$

65. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent p-series with $p = 1.1 > 1$. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$f(x) = \frac{1}{x \ln x}$ is positive, continuous, and decreasing for $x \ge 2$.

$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \left[\ln|\ln x|\right]_2^{\infty} = \infty$

(b) $\sum_{n=2}^{6} \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$\sum_{n=2}^{6} \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$

For $n \geq 4$, the terms of the convergent series **seem** to be larger than those of the divergent series.

(c) $\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$

$n \ln n < n^{1.1}$

$\ln n < n^{0.1}$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < \left(e^{40}\right)^{0.1} = e^4 \approx 55$.

66. (a) $\int_{10}^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{10}^{\infty} = \frac{1}{(p-1)10^{p-1}},\ p > 1$

(b) $f(x) = \frac{1}{x^p}$

$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p}$

$\leq$ Area under the graph of f over the interval $[10, \infty)$

(c) The horizontal asymptote is $y = 0$. As n increases, the error decreases.

67. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.

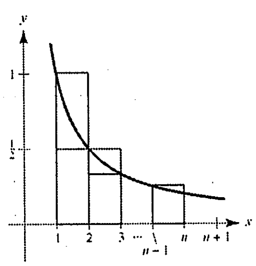
$S_n - 1 \leq \int_1^n \frac{1}{x}\,dx$

$S_n - 1 \leq \ln n$

So, $S_n \leq 1 + \ln n$. Similarly,

$S_n \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$.

So, $\ln(n+1) \leq S_n \leq 1 + \ln n$.



(b) Because $\ln(n+1) \leq S_n \leq 1 + \ln n$, you have $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$. Also, because $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. So, $0 \leq S_n - \ln n \leq 1$ and the sequence $\{a_n\}$ is bounded.

(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \geq 0$

So, $a_n \geq a_{n+1}$ and the sequence is decreasing.

(d) Because the sequence is bounded and monotonic, it converges to a limit, γ.

(e) $a_{100} = S_{100} - \ln 100 \approx 0.5822$ (Actually $\gamma \approx 0.577216$.)

68. $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2 - 1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n + 1)(n - 1)}{n^2} = \sum_{n=2}^{\infty}\left[\ln(n + 1) + \ln(n - 1) - 2 \ln n\right]$

$$= \left(\cancel{\ln 3} + \cancel{\ln 1} - 2 \ln 2\right) + \left(\cancel{\ln 4} + \ln 2 - \cancel{2 \ln 3}\right) + \left(\cancel{\ln 5} + \cancel{\ln 3} - \cancel{2 \ln 4}\right) + \left(\cancel{\ln 6} + \cancel{\ln 4} - \cancel{2 \ln 5}\right)$$
$$+ \left(\cancel{\ln 7} + \cancel{\ln 5} - \cancel{2 \ln 6}\right) + \left(\cancel{\ln 8} + \cancel{\ln 6} - \cancel{2 \ln 7}\right) + \left(\cancel{\ln 9} + \cancel{\ln 7} - \cancel{2 \ln 8}\right) + \cdots = -\ln 2$$

69. $\sum_{n=2}^{\infty} x^{\ln n}$

(a) $x = 1$: $\sum_{n=2}^{\infty} 1^{\ln n} = \sum_{n=2}^{\infty} 1$, diverges

(b) $x = \frac{1}{e}$: $\sum_{n=2}^{\infty}\left(\frac{1}{e}\right)^{\ln n} = \sum_{n=2}^{\infty} e^{-\ln n} = \sum_{n=2}^{\infty} \frac{1}{n}$, diverges

(c) Let x be given, $x > 0$. Put $x = e^{-p} \Leftrightarrow \ln x = -p$.

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} e^{-p \ln n} = \sum_{n=2}^{\infty} n^{-p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$$

This series converges for $p > 1 \Rightarrow x < \frac{1}{e}$.

70. $\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Converges for $x > 1$ by Theorem 9.11

71. Let $f(x) = \frac{1}{3x - 2}$, $f'(x) = \frac{-3}{(3x - 2)^2} < 0$ for $x \ge 1$

f is positive, continuous, and decreasing for $x \ge 1$.

$$\int_1^{\infty} \frac{1}{3x - 2}\,dx = \left[\frac{1}{3}\ln|3x - 2|\right]_1^{\infty} = \infty$$

So, the series $\sum_{n=1}^{\infty} \frac{1}{3n - 2}$

diverges by Theorem 9.10.

72. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

Let $f(x) = \frac{1}{x\sqrt{x^2 - 1}}$.

f is positive, continuous, and decreasing for $x \ge 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2 - 1}}\,dx = \left[\operatorname{arcsec} x\right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 9.10

73. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

p-series with $p = \frac{5}{4}$

Converges by Theorem 9.11

74. $3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

p-series with $p = 0.95$

Diverges by Theorem 9.11

75. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 9.6

76. $\sum_{n=0}^{\infty} (1.042)^n$ is geometric with $r = 1.042 > 1$. Diverges by Theorem 9.6.

77. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + (1/n^2)}} = 1 \ne 0$$

Diverges by Theorem 9.9

78. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$

Because these are both convergent p-series, the difference is convergent.

79. $\sum_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^n$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \ne 0$$

Fails nth-Term Test

Diverges by Theorem 9.9