9.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using the Integral Test In Exercises 1–22, confirm that the Integral Test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series.

1. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

2. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$

3. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

4. $\sum_{n=1}^{\infty} 3^{-n}$

5. $\sum_{n=1}^{\infty} e^{-n}$

6. $\sum_{n=1}^{\infty} ne^{-n/2}$

7. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \cdots$

8. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

9. $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \frac{\ln 5}{5} + \frac{\ln 6}{6} + \cdots$

10. $\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \frac{\ln 5}{\sqrt{5}} + \frac{\ln 6}{\sqrt{6}} + \cdots$

11. $\frac{1}{\sqrt{1}(\sqrt{1}+1)} + \frac{1}{\sqrt{2}(\sqrt{2}+1)} + \frac{1}{\sqrt{3}(\sqrt{3}+1)} + \cdots + \frac{1}{\sqrt{n}(\sqrt{n}+1)} + \cdots$

12. $\frac{1}{4} + \frac{2}{7} + \frac{3}{12} + \cdots + \frac{n}{n^2+3} + \cdots$

13. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

14. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

15. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

16. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

17. $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$

18. $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

19. $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$

20. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

21. $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

22. $\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1}$

Using the Integral Test In Exercises 23 and 24, use the Integral Test to determine the convergence or divergence of the series, where k is a positive integer.

23. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$

24. $\sum_{n=1}^{\infty} n^k e^{-n}$

Requirements of the Integral Test In Exercises 25–28, explain why the Integral Test does not apply to the series.

25. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

26. $\sum_{n=1}^{\infty} e^{-n} \cos n$

27. $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$

28. $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

Using the Integral Test In Exercises 29–32, use the Integral Test to determine the convergence or divergence of the p-series.

29. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

30. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

31. $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

32. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Using a p-Series In Exercises 33–38, use Theorem 9.11 to determine the convergence or divergence of the p-series.

33. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

34. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

35. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

36. $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$

37. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

38. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

39. **Numerical and Graphical Analysis** Use a graphing utility to find the indicated partial sum S_n and complete the table. Then use a graphing utility to graph the first 10 terms of the sequence of partial sums. For each series, compare the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

(a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^{n-1} = \frac{15}{4}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

40. **Numerical Reasoning** Because the harmonic series diverges, it follows that for any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=1}^{N} \frac{1}{n} > M.$$

(a) Use a graphing utility to complete the table.

M	2	4	6	8
N				

(b) As the real number M increases in equal increments, does the number N increase in equal increments? Explain.

WRITING ABOUT CONCEPTS

41. Integral Test State the Integral Test and give an example of its use.

42. *p*-Series Define a *p*-series and state the requirements for its convergence.

43. Using a Series A friend in your calculus class tells you that the following series converges because the terms are very small and approach 0 rapidly. Is your friend correct? Explain.

$$\frac{1}{10{,}000} + \frac{1}{10{,}001} + \frac{1}{10{,}002} + \cdots$$

44. Using a Function Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Use a graph to rank the following quantities in decreasing order. Explain your reasoning.

(a) $\sum_{n=2}^{7} a_n$ (b) $\int_1^7 f(x)\,dx$ (c) $\sum_{n=1}^{6} a_n$

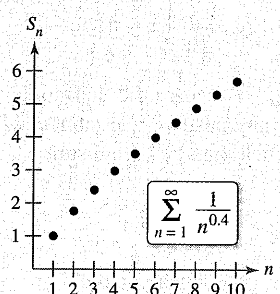
45. Using a Series Use a graph to show that the inequality is true. What can you conclude about the convergence or divergence of the series? Explain.

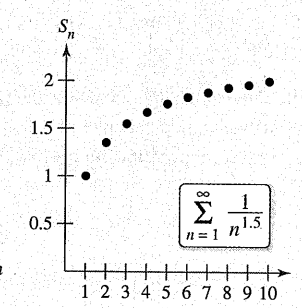
(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ (b) $\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$

46. HOW DO YOU SEE IT? The graphs show the sequences of partial sums of the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.4}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}.$$

Using Theorem 9.11, the first series diverges and the second series converges. Explain how the graphs show this.





Finding Values **In Exercises 47–52, find the positive values of p for which the series converges.**

47. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

48. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

49. $\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

50. $\sum_{n=1}^{\infty} n(1+n^2)^p$

51. $\sum_{n=1}^{\infty} \left(\frac{3}{p}\right)^n$

52. $\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$

53. Proof Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Prove that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges to S, then the remainder $R_N = S - S_N$ is bounded by

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

54. Using a Remainder Show that the result of Exercise 53 can be written as

$$\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx.$$

Approximating a Sum **In Exercises 55–60, use the result of Exercise 53 to approximate the sum of the convergent series using the indicated number of terms. Include an estimate of the maximum error for your approximation.**

55. $\sum_{n=1}^{\infty} \frac{1}{n^2}$, five terms

56. $\sum_{n=1}^{\infty} \frac{1}{n^5}$, six terms

57. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, ten terms

58. $\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3}$, ten terms

59. $\sum_{n=1}^{\infty} ne^{-n^2}$, four terms

60. $\sum_{n=1}^{\infty} e^{-n}$, four terms

Finding a Value **In Exercises 61–64, use the result of Exercise 53 to find N such that $R_N \leq 0.001$ for the convergent series.**

61. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

62. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

63. $\sum_{n=1}^{\infty} e^{-n/2}$

64. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

65. Comparing Series

(a) Show that $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) Compare the first five terms of each series in part (a).

(c) Find $n > 3$ such that $\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$.

66. Using a *p*-Series Ten terms are used to approximate a convergent *p*-series. Therefore, the remainder is a function of p and is

$$0 \leq R_{10}(p) \leq \int_{10}^{\infty} \frac{1}{x^p}\,dx, \quad p > 1.$$

(a) Perform the integration in the inequality.

(b) Use a graphing utility to represent the inequality graphically.

(c) Identify any asymptotes of the error function and interpret their meaning.

67. Euler's Constant Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(a) Show that $\ln(n+1) \le S_n \le 1 + \ln n$.

(b) Show that the sequence $\{a_n\} = \{S_n - \ln n\}$ is bounded.

(c) Show that the sequence $\{a_n\}$ is decreasing.

(d) Show that a_n converges to a limit γ (called Euler's constant).

(e) Approximate γ using a_{100}.

68. Finding a Sum Find the sum of the series

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right).$$

69. Using a Series Consider the series $\sum_{n=2}^{\infty} x^{\ln n}$.

(a) Determine the convergence or divergence of the series for $x = 1$.

(b) Determine the convergence or divergence of the series for $x = 1/e$.

(c) Find the positive values of x for which the series converges.

70. Riemann Zeta Function The **Riemann zeta function** for real numbers is defined for all x for which the series

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$$

converges. Find the domain of the function.

Review **In Exercises 71–82, determine the convergence or divergence of the series.**

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-2}$

72. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$

74. $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

75. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

76. $\displaystyle\sum_{n=0}^{\infty} (1.042)^n$

77. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

78. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right)$

79. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

80. $\displaystyle\sum_{n=2}^{\infty} \ln n$

81. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

82. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

SECTION PROJECT

The Harmonic Series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is one of the most important series in this chapter. Even though its terms tend to zero as n increases,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

the harmonic series diverges. In other words, even though the terms are getting smaller and smaller, the sum "adds up to infinity."

(a) One way to show that the harmonic series diverges is attributed to James Bernoulli. He grouped the terms of the harmonic series as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{> \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{> \frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{> \frac{1}{2}} + \cdots$$

Write a short paragraph explaining how you can use this grouping to show that the harmonic series diverges.

(b) Use the proof of the Integral Test, Theorem 9.10, to show that

$$\ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n.$$

(c) Use part (b) to determine how many terms M you would need so that

$$\sum_{n=1}^{M} \frac{1}{n} > 50.$$

(d) Show that the sum of the first million terms of the harmonic series is less than 15.

(e) Show that the following inequalities are valid.

$$\ln\frac{21}{10} \le \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{20} \le \ln\frac{20}{9}$$

$$\ln\frac{201}{100} \le \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} \le \ln\frac{200}{99}$$

(f) Use the inequalities in part (e) to find the limit

$$\lim_{m\to\infty} \sum_{n=m}^{2m} \frac{1}{n}.$$