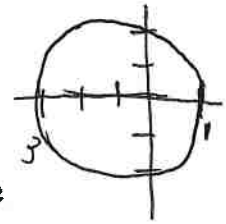
Key

9.4 AP Practice Problems (p.674) – Polar Area



1. The area in the second quadrant bounded by the graph of the polar equation $r = 2 - \cos\theta$ is

(A) π (B) $\frac{9}{4}\pi + 4$ (C) $\frac{5}{8}\pi + 2$ $\boxed{\text{(D)}}$ $\frac{9}{8}\pi + 2$

$\pi/2 < \theta < \pi$

$$\text{Area} = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$A = \frac{1}{2}\int_{\pi/2}^{\pi} [2-\cos\theta]^2\,d\theta$$

$$\frac{1}{2}\int 4 - 4\cos\theta + \cos^2\theta$$

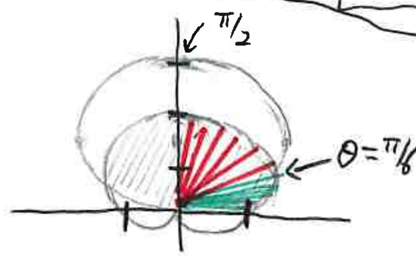
$$*\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\frac{1}{2}\left(4\theta - 4\sin\theta + \frac{1}{2}\theta + \frac{1}{2}\sin(2\theta)\right)\Big]_{\pi/2}^{\pi}$$

$$2\pi - \cancel{2\sin\pi} + \frac{1}{4}\pi + \cancel{\frac{1}{4}\sin 2\pi} - \left(2\cdot\frac{\pi}{2} - 2(1) + \frac{1}{4}\left(\frac{\pi}{2}\right)\right) + \frac{1}{4}(0)$$

$$2\pi + \frac{\pi}{4} - \pi + 2 - \frac{\pi}{8} = \boxed{\frac{9\pi}{8} + 2}$$

2. Find the area of the region common to the graphs of $r = 1 + \sin\theta$ and $r = 3\sin\theta$.



(A) $2\left[\frac{1}{2}\int_0^{\pi/2}(3\sin\theta)^2 d\theta + \frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta\right]$

(B) $2\left[\frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/2}(3\sin\theta)^2 d\theta\right]$

(C) $2\left[\frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2 d\theta - \int_0^{\pi/6}(3\sin\theta)^2 d\theta\right]$

$\boxed{\text{(D)}}$ $2\left[\frac{1}{2}\int_0^{\pi/6}(3\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta\right]$

* Intersection:

$1 + \sin\theta = 3\sin\theta$

$1 = 2\sin\theta$

$\frac{1}{2} = \sin\theta$

$\theta = \pi/6,\ 5\pi/6$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/6}(3\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}[1+\sin\theta]^2 d\theta\right]$$

3. The graph of the polar equation $r = \cos(2\theta)$ is a rose with four petals. Find the area of one petal.

(A) $\frac{\pi}{16}$ $\boxed{\text{(B)}}$ $\frac{\pi}{8}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

$\frac{1}{2}\cdot\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$

$\frac{\pi}{8}+\frac{1}{8}\sin\pi-(0-0)$

$=\frac{\pi}{8}+0=\boxed{\pi/8}$

*set r=0 to find a polar zero

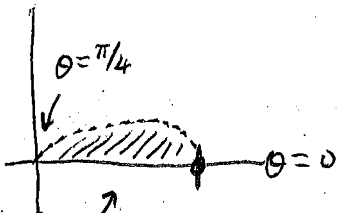
$\cos(2\theta)=r$

$\cos(2\theta)=0$

$2\theta=\cos^{-1}(0)$

$\frac{1}{2}\left[2\theta=\frac{\pi}{2},\frac{3\pi}{2}\dots\right]$

$\theta=\pi/4, 3\pi/4$



half of one petal

$A=\frac{1}{2}\int r^2\,d\theta$

$A=2\cdot\frac{1}{2}\int_0^{\pi/4}(\cos 2\theta)^2\,d\theta$

$\cos^2(2\theta)=\frac{1}{2}(1+\cos 2(2\theta))$

$\int\frac{1}{2}+\frac{1}{2}\cos 4\theta \rightarrow \frac{1}{2}\int 1+\cos 4\theta$

4. The area of the inner loop of the limaçon $r = 2 - 4\sin\theta$ equals

I. $2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}(2-4\sin\theta)^2\,d\theta$

II. $\frac{1}{2}\int_{\pi/6}^{5\pi/6}(2-4\sin\theta)^2\,d\theta$

III. $2\cdot\frac{1}{2}\int_{\pi/2}^{5\pi/6}(2-4\sin\theta)^2\,d\theta$

(A) I and II only (B) I and III only

(C) II and III only $\boxed{\text{(D)}}$ I, II, and III

$r=2-4\sin\theta$

*find polar zeros

$0=2-4\sin\theta$

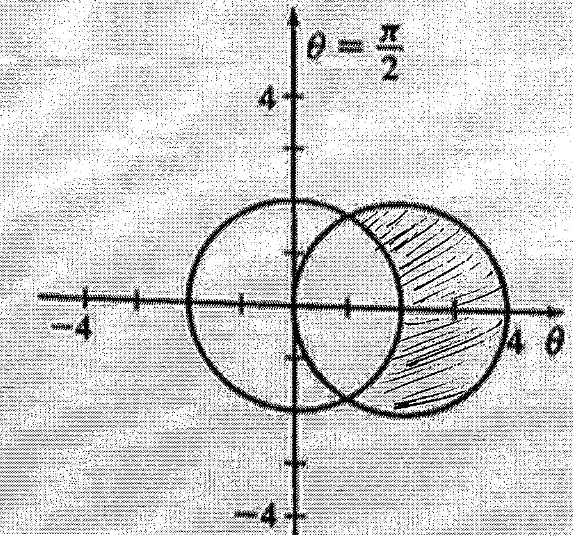
$4\sin\theta=2$

$\sin\theta=1/2$

$\theta=\pi/6, 5\pi/6$

$A=\frac{1}{2}\int_{\pi/6}^{5\pi/6}[2-4\sin\theta]^2\,d\theta$

5. The graphs of the polar equations $r = 2$ and $r = 4\cos\theta$ are shown below.



(a) Find the points of intersection of the two graphs.
(b) Find the area of the region that lies outside of the circle $r = 2$ and inside the circle $r = 4\cos\theta$.

a) $r=2$ and $r=4\cos\theta$ (set equal)

$4\cos\theta = 2$

$\cos\theta = 1/2$

$\theta = \pi/3, 5\pi/3$

$\boxed{\theta = \pi/3, -\pi/3}$

$\boxed{\text{points: } (2, \pi/3) \text{ and } (2, -\pi/3)}$

b) half the shaded region area is $0 \le \theta \le \pi/3$

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/3}[4\cos\theta]^2 d\theta - \frac{1}{2}\int_0^{\pi/3}(2)^2 d\theta\right]$$

$$\int 16\cos^2\theta - \int 4 d\theta$$

$$\int 16\left(\tfrac{1}{2}(1+\cos 2\theta)\right) - 4\, d\theta$$

$$\int 8 + 8\cos 2\theta - 4\, d\theta$$

$$\int 8\cos 2\theta + 4\, d\theta$$

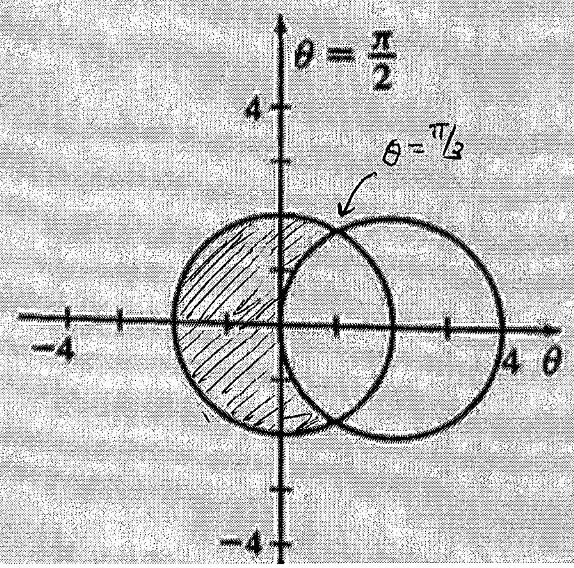
$$8\cdot\tfrac{1}{2}\sin 2\theta + 4\theta\Big]_0^{\pi/3}$$

$$4\sin(2\pi/3) + 4\pi/3 - (0+0)$$

$$4\cdot\frac{\sqrt{3}}{2} + \frac{4\pi}{3}$$

$$\boxed{\text{Area} = \frac{4\pi}{3} + 2\sqrt{3}}$$

5. The graphs of the polar equations $r = 2$ and $r = 4\cos\theta$ are shown below.



(c) Find the area of the region that lies inside the circle $r = 2$ but outside of the circle $r = 4\cos\theta$.

*find intersection: $4\cos\theta = 2$

$\cos\theta = 1/2 \rightarrow \theta = \pi/3,\ 5\pi/3$

$$\text{Area(sliver in Q1)} = \frac{1}{2}\int_{\pi/3}^{\pi/2} 2^2\,d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\,d\theta$$

$$\frac{1}{2}\int 4 - 16\cos^2\theta\,d\theta$$

$$\int 2 - 8\left(\tfrac{1}{2}(1+\cos 2\theta)\right)d\theta$$

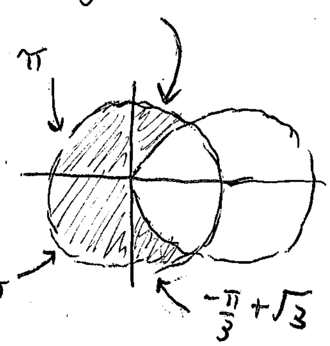
$$2\theta - 4\theta - 4\cdot\tfrac{1}{2}\sin 2\theta$$

$$-2\theta - 2\sin 2\theta\Big]_{\pi/3}^{\pi/2}$$

$$-2(\pi/2) - 2\sin\pi - \left(-\frac{2\pi}{3} - 2\sin\frac{2\pi}{3}\right)$$

$$-\pi - 0 + \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)$$

$$-\frac{\pi}{3} + \sqrt{3}$$



$$\pi + \pi - \frac{\pi}{3} + \sqrt{3} - \frac{\pi}{3} + \sqrt{3}$$

$$\boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$