AP Calculus BC 9.4 Notes Direct Comparison Test and Limit Comparison Test

-Use the direct comparison test to determine convergence.

The convergence tests so far (nth-term, geometric, integral, and p-series) have been fairly simple and the series have special characteristics that make finding convergence easy. Any slight deviation from those characteristics can yield a series where the previous tests would not apply. For example,

1) $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.

2) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.

3) $a_n = \frac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2+3)^2}$ is not.

The direct comparison test is a tool that we can use to determine convergence for complicated, positive series by comparing them with simpler series.

I. Direct Comparison Test (DCT)

Let $0 < a_n \le b_n$ for all n.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note: You must state/show the inequality when stating the conclusion of this test.

Example 1: Determine Convergence or Divergence

(a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

(d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

(e) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

g) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

h) $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

Often a given series closely resembles a p-series, but doesn't exactly match up term-by-term to apply the Direct Comparison Test. If this is the case, there is a second comparison test called the **Limit Comparison Test.**

II. Limit Comparison Test(LCT)

Suppose that $a_n > 0$, $b_n > 0$, and

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$ **OR** $\lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = L$ where L is ***finite** and **positive***. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(further clarification: if L is finite and positive, then L cannot equal zero and L cannot equal infinity)

Example 2: Using the Limit Comparison Test

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{an+b}\right),\ a>0,\ b>0$$

The Limit Comparison Test works well when comparing complicated algebraic series with a p-series. To choose the appropriate p-series, make it match up with the highest power of the complicated series.

Example 3:

$$\sum_{n=1}^{\infty}\frac{1}{3n^2-4n+5}$$

choose $\sum_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$$

choose $\sum_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty}\frac{n^2-10}{4n^5+n^3}$$

choose $\sum_{n=1}^{\infty}$

Example 4: Determine the convergence or divergence of the following series

a) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

b) $\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$

c) $\sum_{n=1}^{\infty}\frac{n^4+10}{4n^5-n^3+7}$

* Comparison of Series

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-Use the direct comparison test to determine convergence.

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1) $\sum_{n=0}^{\infty}\frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty}\frac{n}{2^n}$ is not.

2) $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$ is not.

3) $a_n=\frac{n}{(n^2+3)^2}$ is easily integrated, but $b_n=\frac{n^2}{(n^2+3)^2}$ is not.

The direct comparison test is a tool that we can use to determine convergence for complicated, positive series by comparing them with simpler series.

I. Direct Comparison Test (DCT)

If b_n converges, then anything smaller (a_n) will converge as well.

If a series diverges (a_n), then any series that is larger (b_n) will diverge as well.

Let $0 < a_n \le b_n$ for all n.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note: You must state/show the inequality when stating the conclusion of this test.

$0 \le a_n \le b_n$

Example 1: Determine Convergence or Divergence

(a) $\sum_{n=1}^{\infty}\frac{n^3}{n^3+1}$

n^{th} term test

$\lim_{n\to\infty}\frac{n^3}{n^3+1}=1\neq 0$

By n^{th} term test, series diverges.

(b) $\sum_{n=1}^{\infty}\frac{1}{n^3+1}$

Let $b_n=\frac{1}{n^3}$

Let $a_n=\frac{1}{n^3+1}$

Since $\frac{1}{n^3}$ converges by p-series test, and $a_n<b_n$, $\frac{1}{n^3+1}$ also converges by Direct Comparison Test.

(c) $\sum_{n=1}^{\infty}\frac{1}{3^n+2}$

Since $\left(\frac{1}{3}\right)^n>\frac{1}{3^n+2}$ and $\left(\frac{1}{3}\right)^n$ converges by geometric series test, since $r=\frac{1}{3}<1$, $\frac{1}{3^n+2}$ converges by DCT.

(d) $\sum_{n=4}^{\infty}\frac{1}{\sqrt{n}-1}$

Since $\frac{1}{\sqrt{n}}<\frac{1}{\sqrt{n}-1}$ and $\frac{1}{n^{1/2}}$ diverges by p-series test since $p=\frac{1}{2}<1$, $\frac{1}{\sqrt{n}-1}$ also diverges by DCT.

(e) $\sum_{n=1}^{\infty}\frac{\cos n}{2^n}$

$-1\le\cos n\le 1$

$\boxed{\frac{|\cos n|}{2^n}\le\frac{1}{2^n}}$

Since $\frac{|\cos n|}{2^n}\le\frac{1}{2^n}$, and $\left(\frac{1}{2}\right)^n$ is geometric series and converges since $r=\frac{1}{2}<1$ then $\frac{\cos n}{2^n}$ converges by DCT.

(f) $\sum_{n=2}^{\infty}\frac{1}{n^4-10}$

$\frac{1}{n^4-10}>\frac{1}{n^4}$ ... DCT is not best test. Try using LCT.

$\lim_{n\to\infty}\frac{\frac{1}{n^4-10}}{\frac{1}{n^4}}=\lim_{n\to\infty}\frac{n^4}{n^4-10}=1$

By LCT, series converges as well.

g) $\sum_{n=1}^{\infty}\frac{1}{2+3^n}$

$\frac{1}{2+3^n}\le\frac{1}{3^n}$

Since $\left(\frac{1}{3}\right)^n$ is a geometric series and $r=\frac{1}{3}<1$, converges, so by DCT, series converge as well.

h) $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$

By LCT, $\frac{\frac{1}{2+\sqrt{n}}}{\frac{1}{\sqrt{n}}}$, $\lim_{n\to\infty}\frac{\sqrt{n}}{2+\sqrt{n}}=1$. Since $\frac{1}{\sqrt{n}}$ diverges, then series diverge as well.

Often a given series closely resembles a p-series, but doesn't exactly match up term-by-term to apply the Direct Comparison Test. If this is the case, there is a second comparison test called the Limit Comparison Test.

II. Limit Comparison Test(LCT)

Suppose that $a_n > 0$, $b_n > 0$, and

$\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$ **OR** $\lim\limits_{n\to\infty}\left(\frac{b_n}{a_n}\right) = L$ where L is ***finite** and **positive***. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(further clarification: if L is finite and positive, then L cannot equal zero and L cannot equal infinity)

Example 2: Using the Limit Comparison Test

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{an+b}\right),\ a>0,\ b>0$$ compare to $\frac{1}{n}$ (divergent)

$$\lim_{n\to\infty}\frac{\frac{1}{an+b}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{an+b} = \frac{1}{a}$$ is finite, positive

since $\frac{1}{n}$ is divergent, by Limit Comparison Test (LCT), $\frac{1}{an+b}$ diverges as well.

The Limit Comparison Test works well when comparing complicated algebraic series with a p-series. To choose the appropriate p-series, make it match up with the highest power of the complicated series.

Example 3:

n^{th} term test inconclusive

$$\sum_{n=1}^{\infty}\frac{1}{3n^2-4n+5}$$

choose $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{\frac{1}{3n^2-4n+5}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{3n^2-4n+5} = \frac{1}{3}$$

Since $\frac{1}{n^2}$ converges, then by LCT, series converge as well.

n^{th} term test inconclusive

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$$

choose $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (divergent series)

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{3n-2}} = \frac{1}{\sqrt{3}}$$

Since $\frac{1}{\sqrt{n}}$ diverges, then by LCT, series ~~converges~~ diverge as well.

$$\sum_{n=1}^{\infty}\frac{n^2-10}{4n^5+n^3}$$

choose $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$

$$\lim_{n\to\infty}\frac{\frac{n^2-10}{4n^5+n^3}}{\frac{1}{n^3}} \quad \lim_{n\to\infty}\frac{n^3(n^2-10)}{4n^5+n^3} = \frac{1}{4}$$

Since $\frac{1}{n^3}$ converges, then by LCT, series converge as well.

Example 4: Determine the convergence or divergence of the following series

a) $$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$$

n^{th} term test inconclusive

choose $\frac{1}{n^{3/2}}$ (convergent)

$$\lim_{n\to\infty}\frac{\left(\frac{\sqrt{n}}{n^2+1}\right)}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{n^{3/2}(\sqrt{n})}{n^2+1} = 1$$

$\boxed{\text{By LCT, series converges.}}$

b) $$\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$$

choose $\frac{2^n}{n^2}$ (divergent)

$$\lim_{n\to\infty}\frac{\frac{n2^n}{4n^3+1}}{\frac{2^n}{n^2}} = \frac{n^3\cancel{2^n}}{(4n^3+1)\cancel{2^n}} = \frac{1}{4}$$

$\boxed{\text{By LCT, series diverges}}$

c) $$\sum_{n=1}^{\infty}\frac{n^4+10}{4n^5-n^3+7}$$

use $\frac{1}{n}$ (divergent)

$$\lim_{n\to\infty}\frac{\frac{n^4+10}{4n^5-n^3+7}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^5+10n}{4n^5-n^3+7} = \frac{1}{4}$$

$\boxed{\text{By LCT, series diverges}}$