9.4 Direct Comparison Test and Limit Comparison Test

p. 628 3–36 all

Direct Comparison Test: * Use this test to determine convergence/divergence for more complex, positive series by comparing with simpler series.

let $0 < a_n \le b_n$

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

---

4) $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ Since $\frac{1}{3n^2+2} < \frac{1}{3n^2}$ and $\frac{1}{3n^2}$ converges by p-series test, then $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ also converges.

6) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ Since $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}}$ or $\frac{1}{n^{1/2}}$ diverges by p-series test, then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges by comparison test.

8) $\sum_{n=0}^{\infty} \frac{3^n}{4^n+5}$ Since $\frac{3^n}{4^n+5} < \left(\frac{3}{4}\right)^n$ and $\left(\frac{3}{4}\right)^n$ converges by geometric series then $\sum_{n=0}^{\infty} \frac{3^n}{4^n+5}$ converges by comparison test.

10) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ Since $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$ and $\frac{1}{\sqrt{n^3}}$ or $\frac{1}{n^{3/2}}$ converges by p-series test, then $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by Comparison Test.

12) $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ Since $\frac{1}{4\sqrt[3]{n}-1} > \frac{1}{4\sqrt[3]{n}}$ and $\frac{1}{4\sqrt[3]{n}}$ or $\frac{1}{4n^{2/3}}$ diverges by p-series test, then $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ diverges.

14) $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ Since $\frac{4^n}{3^n-1} > \left(\frac{4}{3}\right)^n$ and $\left(\frac{4}{3}\right)^n$ diverges by geometric series test, then $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ diverges by Comparison Test.

series in the problem

simpler series for test case

Limit Comparison Test * Given $a_n > 0$ and $b_n > 0$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L \text{ and } L \text{ is finite and positive}$$

then both series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

16) $\sum_{n=1}^{\infty} \frac{2}{3^n-5}$ $\quad a_n = \frac{2}{3^n-5} \quad b_n = \frac{1}{3^n}$ ← convergent geometric series

$$\lim_{n\to\infty}\left(\frac{\frac{2}{3^n-5}}{\frac{1}{3^n}}\right) = \lim_{n\to\infty} \frac{2}{3^n-5}\cdot\frac{3^n}{1} = \lim_{n\to\infty}\frac{2\cdot 3^n}{3^n-5} = \frac{\frac{2\cdot 3^n}{3^n}}{\frac{3^n}{3^n}-\underbrace{\frac{5}{3^n}}_{=0}} = \frac{2}{1-0} = \boxed{2}$$

$\boxed{\text{By Limit Comparison Test, } \frac{2}{3^n-5} \text{ converges.}}$

18) $\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$ $\quad a_n = \frac{3}{\sqrt{n^2-4}} \quad b_n = \frac{1}{n}$ ← divergent harmonic series

$$\lim_{n\to\infty}\left(\frac{\frac{3}{\sqrt{n^2-4}}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{3}{\sqrt{n^2-4}}\cdot\frac{\sqrt{n^2}}{1} = \lim_{n\to\infty}\frac{\frac{3\sqrt{n^2}}{\sqrt{n^2}}}{\sqrt{\frac{n^2}{n^2}-\frac{4}{n^2}}} = \frac{3}{\sqrt{1-0}} = \boxed{3}$$

$\boxed{\text{By Limit Comparison, } \frac{3}{\sqrt{n^2-4}} \text{ diverges.}}$

20) $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ $\quad a_n = \frac{5n-3}{n^2-2n+5} \quad b_n = \frac{1}{n}$ ← divergent harmonic series

$$\lim_{n\to\infty}\left(\frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{5n-3}{n^2-2n+5}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{\frac{5n^2}{n^2}-\frac{3n}{n^2}}{\frac{n^2}{n^2}-\frac{2n}{n^2}+\frac{5}{n^2}} = \frac{5}{1} = \boxed{5}$$

$\boxed{\text{By Limit Comparison, } \frac{5n-3}{n^2-2n+5} \text{ diverges.}}$

24) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$ $\quad a_n = \frac{n}{(n+1)2^{n-1}} \quad b_n = \frac{1}{2^{n-1}}$ ← convergent geometric series.

$$\lim_{n\to\infty}\left(\frac{\frac{n}{(n+1)2^{n-1}}}{\frac{1}{2^{n-1}}}\right) = \lim_{n\to\infty}\frac{n}{(n+1)\cancel{2^{n-1}}}\cdot\frac{\cancel{2^{n-1}}}{1} = \boxed{1}$$

$\boxed{\text{By Limit Comparison, } \frac{n}{(n+1)2^{n-1}} \text{ converges}}$

$\ast$ 28) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ $a_n = \tan\left(\frac{1}{n}\right)$ $b_n = \frac{1}{n}$

$\lim_{n\to\infty}\left(\frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right) =$

29) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ diverges using <u>p-series test</u> $p = \frac{1}{2} < 1$

30) $\sum_{n=0}^{\infty} 5\left(\frac{-1}{5}\right)^n$ converges using <u>geometric series test</u> $\left|\frac{-1}{5}\right| < 1$.

31) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ Since $\frac{1}{3^n+2} < \frac{1}{3^n}$ and $\frac{1}{3^n}$ or $\left(\frac{1}{3}\right)^n$ converges by geometric series test

then $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ converges by <u>Direct Comparison Test</u>

32) $\sum_{n=4}^{\infty} \frac{1}{3n^2-2n-15}$ $a_n = \frac{1}{3n^2-2n-15}$ $b_n = \frac{1}{n^2} \leftarrow$ converges by p-series test.

$\lim_{n\to\infty}\left(\frac{\frac{1}{3n^2-2n-15}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{1}{3n^2-2n-15}\cdot\frac{n^2}{1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{3n^2}{n^2}-\frac{2n}{n^2}-\frac{15}{n^2}} = \boxed{\frac{1}{3}}$

By <u>limit comparison test</u>, $\frac{1}{3n^2-2n-15}$ converges.

33) $\sum_{n=1}^{\infty} \frac{n}{2n+3}$ Since $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$, series diverges by <u>$n^{th}$ term test.</u>

34) $\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \ldots - \frac{1}{n+2}\Big)$

$\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$ Converges by <u>Telescoping Series Test</u>

35) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ $f(x)=\frac{x}{(x^2+1)^2}$ $f(x)$ is continuous, decreasing, positive for $x \geq 1$

$\int_1^{\infty} \frac{x}{(x^2+1)^2} dx$ $u=x^2+1$ $\frac{du}{dx}=2x$ $dx=\frac{du}{2x}$ $\int \frac{x}{u^2} \cdot \frac{du}{2x} = \frac{1}{2}\int u^{-2} du = \frac{1}{2}\frac{u^{-1}}{-1}$

$= \frac{-1}{2}\left(\frac{1}{x^2+1}\right)\Big]_1^{\infty} = \frac{-1}{2}\left(\frac{1}{\infty}\right) - \left(\frac{-1}{2}\left(\frac{1}{2}\right)\right)$

$= \boxed{\frac{1}{4}}$

Converges by Integral Test.

36) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ Since $\frac{3}{n^2+3n} < \frac{3}{n^2}$ and $\frac{3}{n^2}$ converges by p-series test, then $\frac{3}{n^2+3n}$ converges by direct comparison test.