9.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

1. Graphical Analysis The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

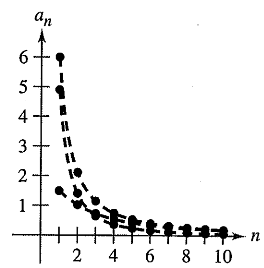
$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}, \quad \sum_{n=1}^{\infty} \frac{6}{n^{3/2}+3}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2+0.5}}$$

(a) Identify the series in each figure.

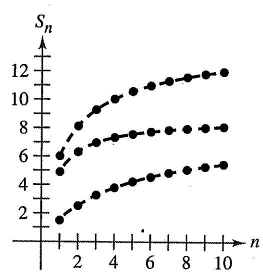
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

2. Graphical Analysis The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

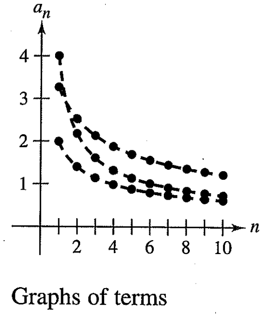
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}-0.5}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4}{\sqrt{n+0.5}}$$

(a) Identify the series in each figure.

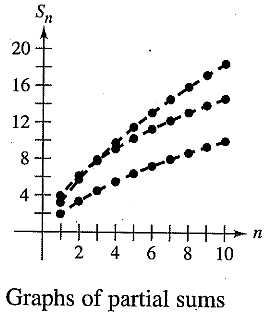
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

Using the Direct Comparison Test **In Exercises 3–12, use the Direct Comparison Test to determine the convergence or divergence of the series.**

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

11. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$

Using the Limit Comparison Test **In Exercises 13–22, use the Limit Comparison Test to determine the convergence or divergence of the series.**

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}, \quad k > 2$

22. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Determining Convergence or Divergence **In Exercises 23–30, test for convergence or divergence, using each test at least once. Identify which test was used.**

(a) *n*th-Term Test

(b) Geometric Series Test

(c) *p*-Series Test

(d) Telescoping Series Test

(e) Integral Test

(f) Direct Comparison Test

(g) Limit Comparison Test

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

24. $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

28. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

31. Using the Limit Comparison Test Use the Limit Comparison Test with the harmonic series to show that the series Σa_n (where $0 < a_n < a_{n-1}$) diverges when $\lim_{n\to\infty} na_n$ is finite and nonzero.

32. **Proof** Prove that, if $P(n)$ and $Q(n)$ are polynomials of degree j and k, respectively, then the series

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges if $j < k - 1$ and diverges if $j \geq k - 1$.

Determining Convergence or Divergence **In Exercises 33–36, use the polynomial test given in Exercise 32 to determine whether the series converges or diverges.**

33. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots$

34. $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

Verifying Divergence **In Exercises 37 and 38, use the divergence test given in Exercise 31 to show that the series diverges.**

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 1}{4n^3 + 2}$

Determining Convergence or Divergence **In Exercises 39–42, determine the convergence or divergence of the series.**

39. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \frac{1}{800} + \cdots$

40. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \frac{1}{230} + \cdots$

41. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \cdots$

42. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots$

WRITING ABOUT CONCEPTS

43. **Using Series** Review the results of Exercises 39–42. Explain why careful analysis is required to determine the convergence or divergence of a series and why only considering the magnitudes of the terms of a series could be misleading.

44. **Direct Comparison Test** State the Direct Comparison Test and give an example of its use.

45. **Limit Comparison Test** State the Limit Comparison Test and give an example of its use.

46. **Comparing Series** It appears that the terms of the series

$$\frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots$$

are less than the corresponding terms of the convergent series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

If the statement above is correct, then the first series converges. Is this correct? Why or why not? Make a statement about how the divergence or convergence of a series is affected by the inclusion or exclusion of the first finite number of terms.

47. **Using a Series** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(a) Verify that the series converges.

(b) Use a graphing utility to complete the table.

n	5	10	20	50	100
S_n					

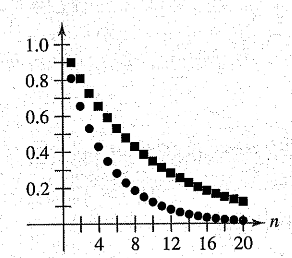
(c) The sum of the series is $\pi^2/8$. Find the sum of the series

$$\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2}.$$

(d) Use a graphing utility to find the sum of the series

$$\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2}.$$

48. **HOW DO YOU SEE IT?** The figure shows the first 20 terms of the convergent series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the first 20 terms of the series $\displaystyle\sum_{n=1}^{\infty} a_n^2$. Identify the two series and explain your reasoning in making the selection.



True or False? **In Exercises 49–54, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

49. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

50. If $0 < a_{n+10} \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

51. If $a_n + b_n \leq c_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge. (Assume that the terms of all three series are positive.)

52. If $a_n \leq b_n + c_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} b_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ both diverge. (Assume that the terms of all three series are positive.)

53. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

54. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

55. Proof Prove that if the nonnegative series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

converge, then so does the series $\sum_{n=1}^{\infty} a_n b_n$.

56. Proof Use the result of Exercise 55 to prove that if the nonnegative series $\sum_{n=1}^{\infty} a_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n^2$.

57. Finding Series Find two series that demonstrate the result of Exercise 55.

58. Finding Series Find two series that demonstrate the result of Exercise 56.

59. Proof Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\Sigma\, b_n$ converges, $\Sigma\, a_n$ also converges.

60. Proof Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\Sigma\, b_n$ diverges, $\Sigma\, a_n$ also diverges.

61. Verifying Convergence Use the result of Exercise 59 to show that each series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$

62. Verifying Divergence Use the result of Exercise 60 to show that each series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

63. Proof Suppose that $\Sigma\, a_n$ is a series with positive terms. Prove that if $\Sigma\, a_n$ converges, then $\Sigma \sin a_n$ also converges.

64. Proof Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n}$$

converges.

65. Comparing Series Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$.

PUTNAM EXAM CHALLENGE

66. Is the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)/n}}$ convergent? Prove your statement.

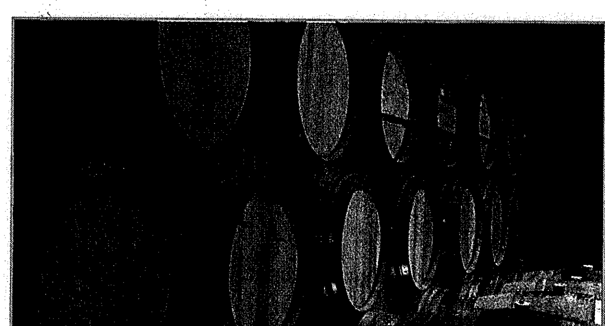
67. Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\displaystyle\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

These problems were composed by the Committee on the Putnam Prize Competition.

SECTION PROJECT

Solera Method

Most wines are produced entirely from grapes grown in a single year. Sherry, however, is a complex mixture of older wines with new wines. This is done with a sequence of barrels (called a solera) stacked on top of each other, as shown in the photo.



The oldest wine is in the bottom tier of barrels, and the newest is in the top tier. Each year, half of each barrel in the bottom tier is bottled as sherry. The bottom barrels are then refilled with the wine from the barrels above. This process is repeated throughout the solera, with new wine being added to the top barrels.

A mathematical model for the amount of n-year-old wine that is removed from a solera (with k tiers) each year is

$$f(n, k) = \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n+1}, \quad k \le n.$$

(a) Consider a solera that has five tiers, numbered $k = 1, 2, 3, 4,$ and 5. In 1995 ($n = 0$), half of each barrel in the top tier (tier 1) was refilled with new wine. How much of this wine was removed from the solera in 1996? In 1997? In 1998? . . . In 2010? During which year(s) was the greatest amount of the 1995 wine removed from the solera?

(b) In part (a), let a_n be the amount of 1995 wine that is removed from the solera in year n. Evaluate

$$\sum_{n=0}^{\infty} a_n.$$

■ **FOR FURTHER INFORMATION** See the article "Finding Vintage Concentrations in a Sherry Solera" by Rhodes Peele and John T. MacQueen in the *UMAP Modules.*