Key

BC Calculus – 9.4a Notes – Defining and Differentiating in Polar Form

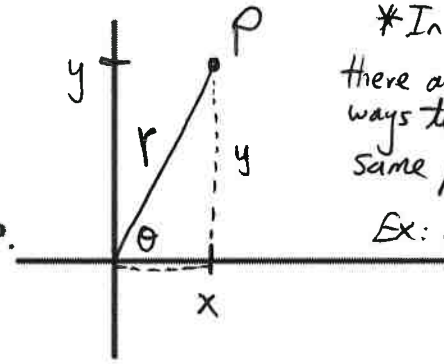
(cartesian)

(x, y) is for a **rectangular** coordinate system.

(r, θ) is for a **polar** coordinate system.

r is a directed distance from the origin to a point P.

θ is the directed angle



*In polar form, there are multiple ways to identify the same point:

Ex: $(3, \frac{\pi}{4}) = (-3, \frac{5\pi}{4}) = (-3, -\frac{3\pi}{4})$

Polar ⟹ Rectangular	Rectangular ⟹ Polar
$x = r\cos\theta$	$\tan\theta = \frac{y}{x}$
$y = r\sin\theta$	$r^2 = x^2 + y^2$

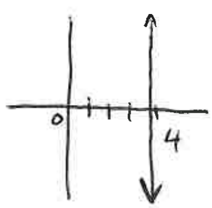
$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

Convert the following from polar form to rectangular form.

1. $\underbrace{r\cos\theta}_{} = -4$

$\boxed{x = -4}$



2. $\underbrace{4r\cos\theta}_{} = \underbrace{r^2}_{}$

$4x = x^2 + y^2$

$0 = x^2 - 4x + y^2$

$x^2 - 4x + \underline{4} + y^2 = 0 + \underline{4}$

$\boxed{(x-2)^2 + y^2 = 4}$

center: $(2, 0)$ $r = 2$

(circle)

3. $\frac{4}{2\cos\theta - \sin\theta} = r$

$4 = r(2\cos\theta - \sin\theta)$

$4 = \underbrace{2r\cos\theta}_{} - \underbrace{r\sin\theta}_{}$

$4 = 2x - y$

$\boxed{y = 2x - 4}$

Slope of a Curve in Polar Form

A curve in polar form is given by $r = f(\theta)$, then its rectangular coordinates are given by $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$. The derivative $\frac{dy}{dx}$ is defined the same way as the derivative of a parametric equation.

Apply Product Rule

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{[f(\theta)\sin\theta]'}{[f(\theta)\cos\theta]'} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

The following is an example of a common problem found on the AP Exam!

4. What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

$\frac{y'(0)}{x'(0)}$

$y = r\sin\theta \rightarrow y = (1+2\sin\theta)\sin\theta$

$x = r\cos\theta \rightarrow x = (1+2\sin\theta)\cos\theta$

$y = \sin\theta + 2\sin^2\theta$

$y' = \cos\theta + 4\sin\theta\cos\theta$

$y'(0) = \cos 0 + 4\sin 0\cos 0$

$y'(0) = 1$

$x = \cos\theta + 2\sin\theta\cos\theta$

$x' = -\sin\theta + 2\cos\theta\cos\theta + 2\sin\theta(-\sin\theta)$

$x' = -\sin 0 + 2\cos^2(0) - 2\sin^2(0)$

$x'(0) = 2$

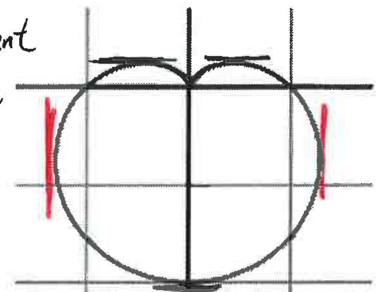
$\frac{y'(0)}{x'(0)} = \boxed{\frac{1}{2}}$

5. Find the value(s) of θ where the polar graph $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$ has horizontal and vertical tangent lines.

*Mode: Polar

*Zoom: 5 Z(square)

slope is $\frac{dy}{dx} \rightarrow \frac{y'(\theta)}{x'(\theta)}$ → set numerator = 0 → horizontal tangent; → set denominator = 0 → vertical tangent



Horizontal:

$y = r\sin\theta$

$y = (1-\sin\theta)\sin\theta$

$y = \sin\theta - \sin^2\theta$

$y' = \cos\theta - 2\sin\theta\cos\theta$

$0 = \cos\theta(1 - 2\sin\theta)$

$\cos\theta = 0$

$\boxed{\cancel{\theta = \frac{\pi}{2}}, \frac{3\pi}{2}}$

$1 - 2\sin\theta = 0$

$\sin\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$

Vertical:

$x = r\cos\theta$

$x = (1-\sin\theta)\cos\theta$

$x' = (-\cos\theta)\cos\theta + (1-\sin\theta)(-\sin\theta)$

$x' = -\cos^2\theta - \sin\theta + \sin^2\theta$

$x' = -(1-\sin^2\theta) - \sin\theta + \sin^2\theta$

$x' = 2\sin^2\theta - \sin\theta - 1$

$0 = (2\sin\theta + 1)(\sin\theta - 1)$

$\sin\theta = -\frac{1}{2}$

$\boxed{\theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$

$\sin\theta = 1$

$\cancel{\theta = \frac{\pi}{2}}$

* Neither vert or horiz. tangent at $\theta = \pi/2$ since cusp exists.

Practice Problems

Problems 1-5 are pre-calculus review on polar form.

1. Find the corresponding rectangular coordinates for the polar coordinates $\left(7, \frac{5\pi}{4}\right)$.

(r, θ)

$X = r\cos\theta$

$X = 7\cos(5\pi/4) = 7\left(-\frac{\sqrt{2}}{2}\right)$

$y = r\sin\theta$

$y = 7\sin(5\pi/4) = 7\left(-\frac{\sqrt{2}}{2}\right)$

$\boxed{\left(-\frac{7\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)}$

2. Calculator active. Find two sets of polar coordinates for the rectangular coordinate $(4, -2)$. Limit your answers on the interval $0 \le \theta \le 2\pi$.

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{-2}{4}$

$\theta = -0.4636 + \pi = 2.678$

OR

$\theta = -0.4636 + 2\pi = 5.819$

$r^2 = x^2 + y^2 \rightarrow r = \sqrt{4^2 + (-2)^2}$

$r = \sqrt{20}$

$\boxed{(\sqrt{20}, 5.819) \text{ or } (\sqrt{20}, 2.678)}$

3. Convert the rectangular equation $x^2 + y^2 = 16$ to a polar equation.

$r^2 = x^2 + y^2$

$r^2 = 16$

$\boxed{r = 4}$

4. Convert the polar equation $r = 3\sec\theta$ to a rectangular equation.

$r = 3\left(\frac{1}{\cos\theta}\right)$

$r = \frac{3}{\cos\theta}$

$r\cos\theta = 3$

* $X = r\cos\theta$

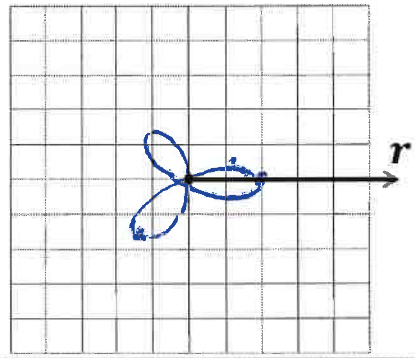
$\boxed{X = 3}$

5. Sketch the polar curve $r = 2\cos 3\theta$ for $0 \le \theta \le \pi$ **without** a calculator, then check your answer.

r	θ
2	0
$\sqrt{2}$	$\pi/12$
0	$\pi/6$
$-\sqrt{2}$	$\pi/4$

r	θ
-2	$\pi/3$
$-\sqrt{2}$	$5\pi/12$
0	$\pi/2$

r	θ
$\sqrt{2}$	$7\pi/12$
2	$2\pi/3$
-2	π



Find the slope of the line tangent to the polar curve at the given value of θ.

6. $r = 3\theta$ at $\theta = \frac{\pi}{2}$.

$x = 3\theta\cos(\theta) \qquad y = 3\theta\sin(\theta)$

$x'(\theta) = 3\cos\theta + 3\theta(-\sin\theta)$

$y'(\theta) = 3\sin\theta + 3\theta\cos\theta$

$r'(\theta) = \dfrac{3\sin\theta + 3\theta\cos\theta}{3\cos\theta - 3\theta\sin\theta}$

$r'(\pi/2) = \dfrac{3\sin(\pi/2) + 3(\pi/2)\cos(\pi/2)}{3\cos(\pi/2) - 3(\pi/2)\sin(\pi/2)}$

$= \dfrac{3+0}{0 - 3(\pi/2)} = \dfrac{3}{-\frac{3\pi}{2}} = 3 \cdot \dfrac{-2}{3\pi} = \boxed{\dfrac{-2}{\pi}}$

7. $r = \frac{5}{3-\cos\theta}$ at $\theta = \frac{3\pi}{2}$.

$x = \dfrac{5}{3-\cos\theta}\cos\theta \qquad y = \dfrac{5}{3-\cos\theta}\cdot\sin\theta$

$x = \dfrac{5\cos\theta}{3-\cos\theta} \qquad y = \dfrac{5\sin\theta}{3-\cos\theta}$

$x'(\theta) = \dfrac{-5\sin\theta(3-\cos\theta) - 5\cos\theta(\sin\theta)}{(3-\cos\theta)^2}$

$y'(\theta) = \dfrac{5\cos\theta(3-\cos\theta) - 5\sin\theta(\sin\theta)}{(3-\cos\theta)^2}$

$x'(3\pi/2) = \dfrac{-5(-1)(3) - 5(0)(-1)}{3^2} = \dfrac{15}{9}$

$y'(3\pi/2) = \dfrac{5(0)(3) - 5(-1)(-1)}{3^2} = \dfrac{-5}{9}$

$r'(3\pi/2) = \dfrac{-5/9}{15/9} = \boxed{\dfrac{-1}{3}}$

$x = r\cos\theta$

8. $r = \cos\theta$ at $\theta = \frac{\pi}{3}$.

$y = r\sin\theta$

$x = \cos\theta\cdot\cos\theta$

$x = \cos^2\theta = [\cos\theta]^2$

$x'(\theta) = 2\cos\theta\cdot -\sin\theta$

$x'(\pi/3) = 2\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\sqrt{3}}{2}$

$y = \cos\theta\cdot\sin\theta$

$y' = -\sin\theta\sin\theta + \cos\theta\cos\theta$

$y'(\pi/3) = -\sin^2(\pi/3) + \cos^2(\pi/3)$

$= -\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= -\frac{3}{4} + \frac{1}{4} = \frac{-2}{4} = -\frac{1}{2}$

$r'(\pi/3) = \dfrac{-1/2}{-\sqrt{3}/2}$

$\boxed{r'(\pi/3) = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}}$

9. $r = 2(1-\sin\theta)$ at $\theta = 0$.

$x = 2(1-\sin\theta)\cos\theta \qquad y = 2(1-\sin\theta)\sin\theta$

$x = 2\cos\theta - 2\sin\theta\cos\theta \qquad y = 2\sin\theta - 2[\sin\theta]^2$

$x'(\theta) = -2\sin\theta - [2\cos\theta\cos\theta + 2\sin\theta(-\sin\theta)]$

$x'(0) = 0 - [2(1) + 0] = -2$

$y'(\theta) = 2\cos\theta - 4\sin\theta\cos\theta$

$y'(0) = 2(1) - 4(0) = 2$

$r'(0) = \dfrac{y'(0)}{x'(0)} = \dfrac{2}{-2} = \boxed{-1}$

10. A particle moves along the polar curve $r = 3\cos\theta$ so that $\frac{d\theta}{dt} = 2$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$. *Hint: remember implicit differentiation?*

Find derivatives with respect to time.

$r = 3\cos\theta$

$\frac{dr}{dt} = -3\sin\theta\left(\frac{d\theta}{dt}\right)$

$\frac{dr}{dt} = -3\sin(\pi/3)(2)$

$\frac{dr}{dt} = -3\left(\frac{\sqrt{3}}{2}\right)(2)$

$\boxed{\frac{dr}{dt} = -3\sqrt{3}}$

11. A polar curve is given by the equation $r = \frac{15\theta}{\theta^2+1}$ for $\theta \geq 0$. What is the instantaneous rate of change of r with respect to θ when $\theta = 1$?

$r'(\theta) = \frac{(15)(\theta^2+1) - 15\theta(2\theta)}{(\theta^2+1)^2}$

$r'(1) = \frac{15(2) - 15(1)(2)(1)}{(1+1)^2} = \frac{30-30}{4} = 0$

$\boxed{r'(1) = 0}$

12. Find the value(s) of θ where the polar graph $r = 2 - 2\cos\theta$ has a horizontal tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

*horizontal tangent when $y' = 0$

$y = r\sin\theta$

$y = (2-2\cos\theta)\sin\theta$

$y' = (2\sin\theta)\sin\theta + (2-2\cos\theta)\cos\theta$

$0 = 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta$

$0 = 2(1-\cos^2(\theta)) + 2\cos\theta - 2\cos^2\theta$

$0 = 2 - 2\cos^2\theta + 2\cos\theta - 2\cos^2\theta$

$0 = -4\cos^2\theta + 2\cos\theta + 2$

$0 = -2(2\cos^2\theta - \cos\theta - 1)$

$0 = -2(2\cos\theta + 1)(\cos\theta - 1)$

$2\cos\theta + 1 = 0$ | $\cos\theta - 1 = 0$

$\cos\theta = -1/2$ | $\cos\theta = 1$

$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$ | $\cancel{\theta = 0, 2\pi}$

this is where $x'(\theta) = 0$ as well

13. Find the value(s) of θ where the polar graph $r = 3 - 3\sin\theta$ has a vertical tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

*vertical tangent when $x' = 0$

$x = (3-3\sin\theta)\cos\theta$

$x'(\theta) = (-3\cos\theta)\cos\theta + (3-3\sin\theta)(-\sin\theta)$

$0 = -3\cos^2\theta - 3\sin\theta + 3[\sin\theta]^2$

$0 = -3(1-\sin^2\theta) - 3\sin\theta + 3\sin^2\theta$

$0 = -3 + 3\sin^2\theta - 3\sin\theta + 3\sin^2\theta$

$0 = 6\sin^2\theta - 3\sin\theta - 3$

$0 = 3(2\sin^2\theta - \sin\theta - 1)$

$0 = 3(2\sin\theta + 1)(\sin\theta - 1)$

$\sin\theta = -1/2$ | $\sin\theta = 1$

$\boxed{\theta = \frac{11\pi}{6}, \frac{7\pi}{6}}$ | $\cancel{\theta = \pi/2}$

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 3$?

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{\sin\theta+\theta\cos\theta}{\cos\theta-\theta\sin\theta}\right]}{\cos\theta - \theta\sin\theta}\Bigg|_{\theta=3} \approx \frac{5.506731}{-1.413352} = \boxed{-3.896}$

Differentiating in Polar Form

15. A polar curve is given by the differentiable function $r = f(\theta)$ for $0 \leq \theta \leq 2\pi$. If the line tangent to the polar curve at $\theta = \frac{\pi}{6}$ is vertical, which of the following must be true?

slope $= \frac{y'}{x'} \to \overline{[r\cos\theta]'} \to \overline{[f(\theta)\cos\theta]'} = 0$

vertical tangent is where $x'(\theta) = 0$

$f'(\theta)\cos\theta + f(\theta)\cdot -\sin\theta = 0$

$f'(\pi/6)\cos(\pi/6) + f(\pi/6)\cdot -\sin(\pi/6) = 0$

$f'(\pi/6)\left(\frac{\sqrt{3}}{2}\right) - f(\pi/6)(1/2) = 0$

A. $f\left(\frac{\pi}{6}\right) = 0$ B. $f'\left(\frac{\pi}{6}\right) = 0$ C. $\frac{1}{2}f\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) = 0$ D. $\boxed{\frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) - \frac{1}{2}f\left(\frac{\pi}{6}\right) = 0}$

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16. **Calculator active.** For $0 \leq t \leq 8$, a particle moving in the xy-plane has position vector $\langle x(t),\ y(t)\rangle = \langle \sin(2t), t^2 - t\rangle$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds.

a. Find the speed of the particle at time $t = 3$ seconds. Indicate units of measure.

*speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$x' = \cos(2t)\cdot 2$ | $y' = 2t - 1$

$\sqrt{[2\cos(2(3))]^2 + [2(3) - 1]^2} \approx \boxed{5.356}$

b. At time $t = 5$ seconds, is the speed of the particle increasing or decreasing? Explain your answer.

$\frac{d}{dt}\sqrt{[2\cos(2t)]^2 + [2t-1]^2}\Big|_{t=5} \approx 1.567$

Increasing speed since velocity and acceleration at $t=5$ is same signs.

c. Find the total distance the particle travels over the time interval $0 \leq t \leq 6$ seconds.

Total Distance $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^6 \sqrt{[2\cos(2t)]^2 + [2t-1]^2}\,dt \approx \boxed{32.436 \text{ meters}}$

d. At time $t = 8$ seconds, the particle begins moving in a straight line. For $t \geq 8$, the particle travels with the same velocity vector that it had at time $t = 8$ seconds. Find the position of the particle at time $t = 11$ seconds.

position at $t=8$: $\langle x(8), y(8)\rangle = \langle -0.2879, 56\rangle$

velocity at $t=8$: $\langle x'(8), y'(8)\rangle = \langle -1.915318, 15\rangle$

3 seconds moving at constant velocity as v(8)

position at $t=11$ is $\langle x(8), y(8)\rangle + 3\langle -1.915318, 15\rangle$

$= \boxed{\langle -6.0338, 101\rangle}$