Key

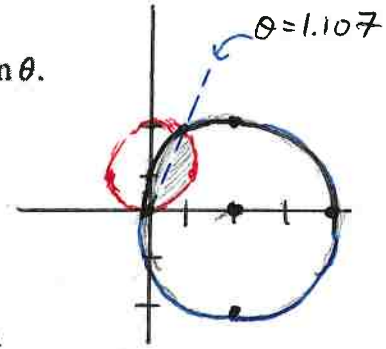
BC Calculus – 9.4c Notes – Area Bounded by two Polar Curves

Recall area bounded by a polar curve: $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$ $\qquad r = f(\theta)$

Things to watch for when using more than one polar curve for area.

- Points of intersection
- Symmetry

1. Find the area of the region common to the polar curve $r = 4\cos\theta$ and $r = 2\sin\theta$.



$4\cos\theta = 2\sin\theta$

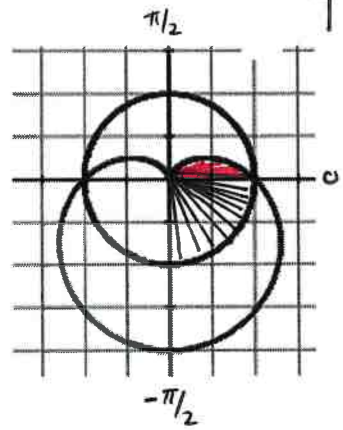
$2 = \tan\theta$

$\theta = 1.107$

$A = \frac{1}{2}\int_0^{1.107} [2\sin\theta]^2 d\theta + \frac{1}{2}\int_{1.107}^{\pi/2} [4\cos\theta]^2 d\theta$

$\boxed{A \approx 0.9617}$

2. Find the area of the common region to the polar graphs of $r = 2$ and $r = 2 - 2\sin\theta$.

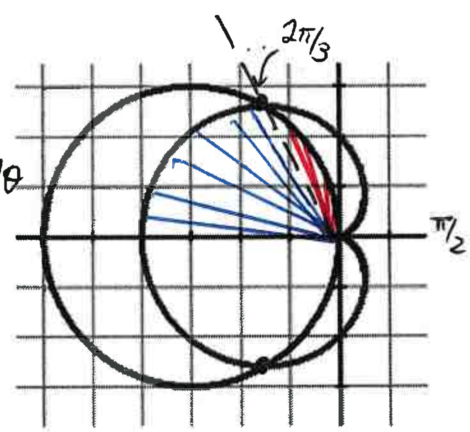


$\frac{1}{2}A = \frac{1}{2}\int_{-\pi/2}^{0} [2]^2 d\theta + \frac{1}{2}\int_0^{\pi/2} [2-2\sin\theta]^2 d\theta$

$\frac{1}{2}A = 3.8539$

$\boxed{A = 7.7079}$

3. Find the area of the region common to the two polar curves $r = -6\cos\theta$ and $r = 2 - 2\cos\theta$.



find intersection:

$-6\cos\theta = 2 - 2\cos\theta$

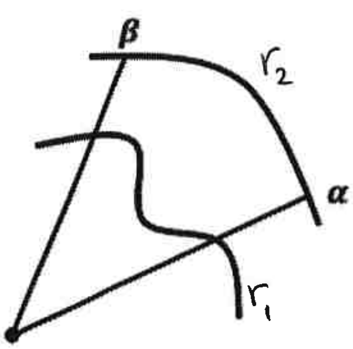
$-4\cos\theta = 2$

$\cos\theta = -1/2$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\frac{1}{2}A = \frac{1}{2}\int_{\pi/2}^{2\pi/3} [-6\cos\theta]^2 d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} [2-2\cos\theta]^2 d\theta$

$\boxed{A = 15.7079}$



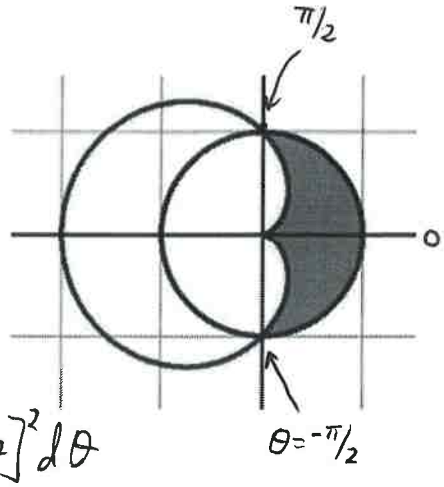
Area Bounded by Two Polar Curves

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2)^2 d\theta - \frac{1}{2}\int_{\alpha}^{\beta} (r_1)^2 d\theta$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta$$

cardioid and circle

4. Find the area of the region bounded by the two polar curves $r = 1$ and $r = 1 - \cos\theta$ as shown in the graph below.



$$\cancel{\tfrac{1}{2}}A = \cancel{\tfrac{1}{2}}\int_0^{\pi/2}[1]^2\,d\theta - \tfrac{1}{2}\int_0^{\pi/2}[1-\cos\theta]^2\,d\theta$$

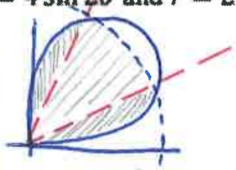
$$\boxed{A = 1.2146}$$

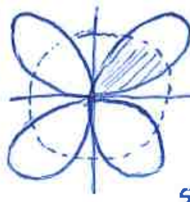
OR $A = \int_{-\pi/2}^{\pi/2}[1]^2 - [1-\cos\theta]^2\,d\theta$

Practice Problems:

find intersection

1. Find the area of the common interior of the polar graphs $r = 4\sin 2\theta$ and $r = 2$.





$4\sin 2\theta = 2$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \sin^{-1}(\frac{1}{2})$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$$A = 4\cdot\left[\frac{1}{2}\int_0^{\pi/12}[4\sin(2\theta)]^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12}[2]^2\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2}[4\sin(2\theta)]^2\,d\theta\right]$$

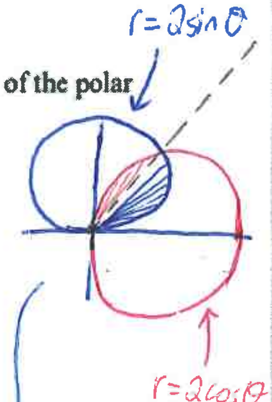
$A = 4(2.4567)$

$$\boxed{A \approx 9.8268}$$

2. Find the area of the common interior of the polar graphs $r = 2\cos\theta$ and $r = 2\sin\theta$.

$2\cos\theta = 2\sin\theta$

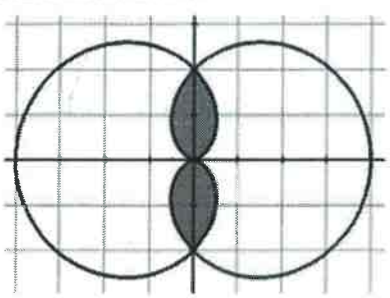
$\theta = \pi/4$



$$\text{Area} = \frac{1}{2}\int_0^{\pi/4}[2\sin\theta]^2\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}[2\cos\theta]^2\,d\theta = \boxed{0.5708}$$

3. The polar curves $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$ are shown below.

$r = 2 - 2\cos\theta$



$A = \frac{1}{2}\int_0^{\pi/2}\left[2[1-\cos\theta]\right]^2$

$4\cdot\frac{1}{2}\cdot 4[1-\cos\theta]^2$

$8\int_0^{\pi/2}[1-\cos\theta]^2\,d\theta$

Which of the following gives the total area of the shaded regions?

A. $\int_0^{\pi}(2 + 2\cos\theta)^2\,d\theta$

B. $\int_{\pi/2}^{\pi}(2 + 2\cos\theta)^2\,d\theta$

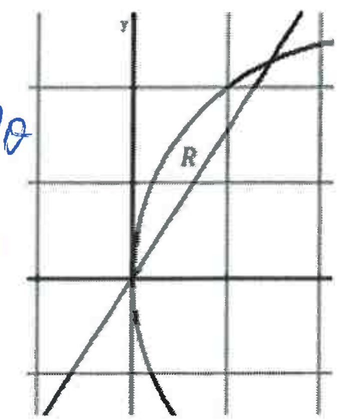
C. $\boxed{8\int_0^{\frac{\pi}{2}}(1 - \cos\theta)^2\,d\theta}$

D. $\int_0^{\frac{\pi}{2}}((2 - 2\cos\theta)^2 + (2 + 2\cos\theta)^2)\,d\theta$

4. Let R be the region in the first quadrant that is bounded above by the polar curve $r = 5\cos\theta$ and below by the line $\theta = 1$, as shown in the figure below. What is the area of R?

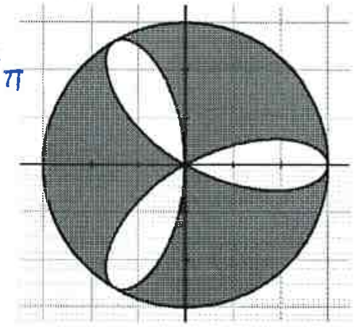
$A = \frac{1}{2}\int_1^{\pi/2}[5\cos\theta]^2\,d\theta$

$$\boxed{0.7259}$$



5. The figure below shows the graphs of the polar curves $r = 3\cos 3\theta$ and $r = 3$. What is the sum of the areas of the shaded regions?

Circle Area $= \pi(3)^2 = 9\pi$



$0 = 3\cos 3\theta$

$3\theta = \cos^{-1}(0)$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \pi/6, 3\pi/6$

$A = \frac{1}{2}\int_{\pi/6}^{\pi/2} [3\cos 3\theta]^2 d\theta$

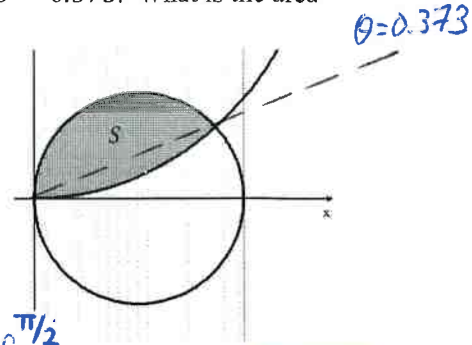
≈ 2.35619

$9\pi - (3 \cdot 2.35619)$

$9\pi - 7.06858$

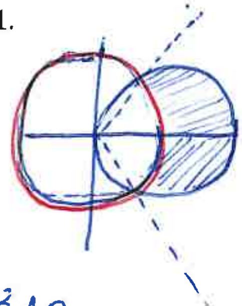
$\approx \boxed{21.2057}$

6. Let S be the region in the 1st Quadrant bounded above by the graph of the polar curve $r = \cos\theta$ and bounded below by the graph of the polar curve $r = \frac{5}{2}\theta$, as shown in the figure above. The two curves intersect when $\theta = 0.373$. What is the area of S?



$A = \frac{1}{2}\int_0^{0.373} [\frac{5}{2}\theta]^2 d\theta + \frac{1}{2}\int_{0.373}^{\pi/2} [\cos\theta]^2 d\theta \approx \boxed{0.2686}$

7. Find the area inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = 1$.



$2\cos\theta = 1$

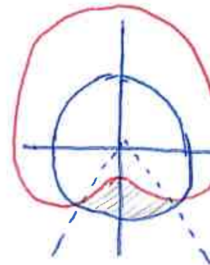
$\cos\theta = 1/2$

$\theta = \frac{\pi}{3}, 5\pi/3 \to \left(\text{or } -\frac{\pi}{3}\right)$

$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} [2\cos\theta]^2 - \frac{1}{2}\int_{-\pi/3}^{\pi/3} [1]^2 d\theta$

$= \boxed{1.913}$

8. Write an integral expression that represents the area of the region outside the polar curve $r = 3 + 2\sin\theta$ and inside the polar curve $r = 2$.



$3 + 2\sin\theta = 2$

$2\sin\theta = -1$

$\sin\theta = -1/2$

$\theta = 11\pi/6, 7\pi/6$

$\boxed{A = \frac{1}{2}\left[\int_{7\pi/6}^{11\pi/6} [2]^2 - [3 + 2\sin\theta]^2 d\theta\right]}$

9. What is the total area outside the polar curve $r = 5\cos 2\theta$ and inside the polar curve $r = 5$?

$5\cos(2\theta) = 0$

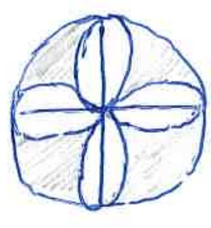
$2\theta = \cos^{-1}(0)$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \pi/4, 3\pi/4$

Area (circle)

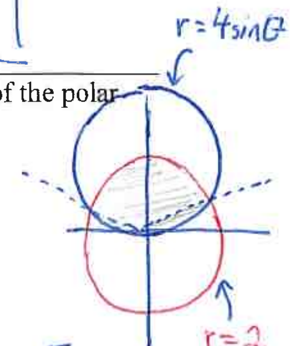
$\pi(5)^2 = 25\pi$



$A = 25\pi - 4 \cdot \frac{1}{2}\int_{\pi/4}^{3\pi/4} [5\cos 2\theta]^2 d\theta$

$25\pi - 4(39.2699) \approx \boxed{39.269}$

10. Find the area of the common interior of the polar curves $r = 4\sin\theta$ and $r = 2$.



$4\sin\theta = 2$

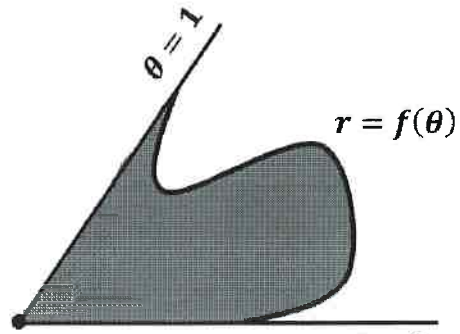
$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

$A = \frac{1}{2}\int_0^{\pi/6} [4\sin\theta]^2 d\theta + \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2]^2 d\theta + \frac{1}{2}\int_{5\pi/6}^{\pi} [4\sin\theta]^2 d\theta$

$\boxed{4.913}$

11.



θ	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
r	1	3	5	4	2

No calculator! Let R be the region bounded by the graph of the polar curve $r = f(\theta)$ and the lines $\theta = 0$ and $\theta = 1$, as shaded in the figure above. The table above gives values of the polar function $r = f(\theta)$ at selected values of θ. What is the approximation for the area of region R using a right Riemann sum with the four subintervals indicated by the data in the table?

*Area of each sector: $\frac{1}{2}\theta r^2$

$$\frac{1}{2}\left(\frac{1}{4}\right)(3)^2 + \frac{1}{2}\left(\frac{1}{4}\right)(5)^2 + \frac{1}{2}\left(\frac{1}{4}\right)(4)^2 + \frac{1}{2}\left(\frac{1}{4}\right)(2)^2 = \frac{54}{8} = \boxed{\frac{27}{4}}$$