66. The series diverges. For $n > 1$,

$$n < 2^n$$
$$n^{1/n} < 2$$
$$\frac{1}{n^{1/n}} > \frac{1}{2}$$
$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}$$

Because $\sum \frac{1}{2n}$ diverges, so does $\sum \frac{1}{n^{(n+1)/n}}$.

67. Consider two cases:

If $a_n \geq \frac{1}{2^{n+1}}$, then $a_n^{1/(n+1)} \geq \left(\frac{1}{2^{n+1}}\right)^{1/(n+1)} = \frac{1}{2}$, and

$$a_n^{n/(n+1)} = \frac{a_n}{a_n^{1/(n+1)}} \leq 2a_n.$$

If $a_n \leq \frac{1}{2^{n+1}}$, then $a_n^{n/(n+1)} \leq \left(\frac{1}{2^{n+1}}\right)^{n/(n+1)} = \frac{1}{2^n}$, and

combining, $a_n^{n/(n+1)} \leq 2a_n + \frac{1}{2^n}$.

Because $\sum_{n=1}^{\infty}\left(2a_n + \frac{1}{2^n}\right)$ converges, so does $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$ by the Comparison Test.

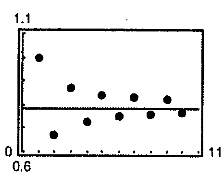
Section 9.5 Alternating Series

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605

(b)



(c) The points alternate sides of the horizontal line $y = \frac{\pi}{4}$ that represents the sum of the series.

The distance between successive points and the line decreases.

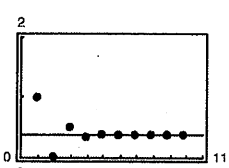
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679

(b)



(c) The points alternate sides of the horizontal line $y = \frac{1}{e}$ that represents the sum of the series.

The distance between successive points and the line decreases.

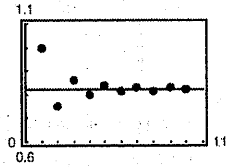
(d) The distance in part (c) is always less than the magnitude of the next series.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180

(b)



(c) The points alternate sides of the horizontal line $y = \frac{\pi^2}{12}$ that represents the sum of the series.

The distance between successive points and the line decreases.

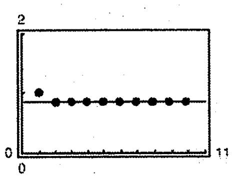
(d) The distance in part (c) is always less than the magnitude of the next term in the series.

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415

(b)



(c) The points alternate sides of the horizontal line $y = \sin(1)$ that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

$a_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

Converges by Theorem 9.14

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$

$\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}$

Diverges by nth-Term test

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

$a_{n+1} = \frac{1}{3^{n+1}} < \frac{1}{3^n} = a_n$

$\lim_{n\to 0} \frac{1}{3^n} = 0$

Converges by Theorem 9.14

(**Note:** $\sum_{n=1}^{\infty} \left(\frac{-1}{3}\right)^n$ is a convergent geometric series)

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

$a_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = a_n$

$\lim_{n\to\infty} \frac{1}{e^n} = 0$

Converges by Theorem 9.14

(Note: $\sum_{n=1}^{\infty} \left(\frac{-1}{e}\right)^n$ is a convergent geometric series)

9. $\sum_{n=1}^{\infty} \frac{(-1)^n(5n-1)}{4n+1}$

$\lim_{n\to\infty} \frac{5n-1}{4n+1} = \frac{5}{4}$

Diverges by nth-Term test

10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$

Let $f(x) = \frac{x}{x^2+5}$, $f'(x) = \frac{-(x^2-5)}{(x^2+5)^2} < 0$ for $x \geq 3$

So, $a_{n+1} < a_n$ for $n \geq 3$

$\lim_{n\to\infty} \frac{n}{n^2+5} = 0$

Converges by Theorem 9.14

11. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$

Diverges by nth-Term test

12. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

Converges by Theorem 9.14

13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges by Theorem 9.14

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

$\lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$

Diverges by nth-Term test

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$

Diverges by the nth-Term Test

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1}$ for $n \geq 2$

$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1/(n+1)}{1} = 0$

Converges by Theorem 9.14

17. $\sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$

Diverges by the nth-Term Test

18. $\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Converges by Theorem 9.14

19. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$

$\lim_{n\to\infty} \frac{1}{n!} = 0$

Converges by Theorem 9.14

20. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$

$\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$

Converges by Theorem 9.14

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2}$ for $n \geq 2$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$

Converges by Theorem 9.14

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

$\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$

Diverges by the nth-Term Test

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$a_{n+1} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n+1}{2n+1} = a_n\left(\frac{n+1}{2n+1}\right) < a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n\to\infty} 2\left[\frac{3}{3} \cdot \frac{4}{5} \cdot \frac{5}{7} \cdots \frac{n}{2n-3}\right] \cdot \frac{1}{2n-1} = 0$

Converges by Theorem 9.14

24. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$

$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} = a_n\left(\frac{2n+1}{3n+1}\right) < a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3\left[\frac{5}{4} \cdot \frac{7}{7} \cdot \frac{9}{10} \cdots \frac{2n-1}{3n-5}\right]\frac{1}{3n-2} = 0$

Converges by Theorem 9.14

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$

Let $f(x) = \frac{2e^x}{e^{2x} - 1}$. Then

$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$

So, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$\lim_{n\to\infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$

The series converges by Theorem 9.14.

26. $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} + 1}$

Let $f(x) = \frac{2e^x}{e^{2x} + 1}$. Then

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

So, $f(x)$ is decreasing for $x > 0$ which implies $a_{n+1} < a_n$.

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 9.14.

27. $S_6 = \sum_{n=0}^{5} \frac{(-1)^n 5}{n!} = \frac{11}{6}$

$|R_6| = |S - S_6| \le a_7 = \frac{5}{720} = \frac{1}{144}$

$\frac{11}{6} - \frac{1}{144} \le S \le \frac{11}{6} + \frac{1}{144}$

$1.8264 \le S \le 1.8403$

28. $S_6 = \sum_{n=1}^{6} \frac{4(-1)^{n+1}}{\ln(n + 1)} \approx 2.7067$

$|R_6| = |S - S_6| \le a_7 = \frac{4}{\ln 8} \approx 1.9236$

$0.7831 \le S \le 4.6303$

29. $S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1} 2}{n^3} \approx 1.7996$

$|R_6| = |S - S_6| \le a_7 = \frac{2}{7^3} \approx 0.0058$

$1.7796 - 0.0058 \le S \le 1.7796 + 0.0058$

$1.7938 \le S \le 1.8054$

30. $S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1} n}{3^n} \approx 0.1852$

$|R_6| = |S - S_6| \le a_7 = \frac{7}{3^7} \approx 0.0032$

$0.1852 - 0.0032 \le S \le 0.1852 + 0.0032$

$0.1820 \le S \le 0.1884$

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

By Theorem 9.15,

$|R_N| \le a_{N+1} = \frac{1}{(N + 1)^3} < 0.001$

$\Rightarrow (N + 1)^3 > 1000 \Rightarrow N + 1 > 10.$

Use 10 terms.

32. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

By Theorem 9.15,

$|R_N| \le a_{N+1} = \frac{1}{(N + 1)^2} < 0.001$

$\Rightarrow (N + 1)^2 > 1000.$

By trial and error, this inequality is valid when $N = 31(32^2 = 1024)$.

Use 31 terms.

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

By Theorem 9.15,

$|R_N| \le a_{N+1} = \frac{1}{2(N + 1)^3 - 1} < 0.001$

$\Rightarrow 2(N + 1)^3 - 1 > 1000.$

By trial and error, this inequality is valid when $N = 7\left[2(8^3) - 1 = 1024\right]$.

Use 7 terms.

34. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

By Theorem 9.15,

$|R_N| \le a_{N+1} = \frac{1}{(N + 1)^5} < 0.001$

$\Rightarrow (N + 1)^5 > 1000.$

By trial and error, this inequality is valid when $N = 3(4^5 = 1024)$.

Use 3 terms.

35. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

By Theorem 9.15,

$|R_N| \le a_{N+1} = \frac{1}{(N + 1)!} < 0.001$

$\Rightarrow (N + 1)! > 1000.$

By trial and error, this inequality is valid when $N = 6(7! = 5040)$. Use 7 terms since the sum begins with $n = 0$.

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

By Theorem 9.15,

$$|R_N| \le a_{N+1} = \frac{1}{(2(N+1))!} = \frac{1}{(2N+2)!} < 0.001$$

$$\Rightarrow (2N+2)! > 1000.$$

By trial and error, this inequality is valid when $N = 3(8! = 40{,}320)$. Use 4 terms since the sum begins with $n = 0$.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely.

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$\displaystyle\frac{1}{n!} < \frac{1}{n^2}$ for $n \ge 4$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges, and

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges absolutely.

40. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

The series converges by the Alternating Series Test. But, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+3}$$

diverges by comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$ converges conditionally.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

The given series converges by the Alternating Series Test, but does not converge absolutely because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p-series. Therefore, the series converges conditionally.

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series.

Therefore, the given series converges absolutely.

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the nth-Term Test.

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$$\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$$

Therefore, the series diverges by the nth-Term Test.

45. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

The series converges by the Alternating Series Test.

Let $f(x) = \dfrac{1}{x \ln x}$.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \Big[\ln(\ln x)\Big]_2^{\infty} = \infty$$

By the Integral Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

So, the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally.

46. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$ converges by a comparison to the convergent geometric series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$. Therefore, the given series converges absolutely.

47. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 5}$

$\sum_{n=2}^{\infty} \frac{n}{n^3 - 5}$ converges by a limit comparison to the p-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Therefore, the given series converges absolutely.

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a convergent p-series. Therefore, the given series converges absolutely.

49. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!}$

$$\sum_{n=0}^{\infty} \frac{1}{(2n + 1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

because

$$\frac{1}{(2n + 1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

50. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n + 4}}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n + 4}}$$

diverges by a limit comparison to the divergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

51. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n + 1} = \sum_{n=0}^{\infty} \frac{1}{n + 1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$\lim_{n \to \infty} \frac{|\cos n\pi|/(n + 1)}{1/n} = 1$, therefore, the series converges conditionally.

52. $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

$$\lim_{n \to \infty} \arctan n = \frac{\pi}{2} \neq 0$$

Therefore, the series diverges by the nth-Term Test.

53. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Therefore, the given series converges absolutely.

54. $\sum_{n=1}^{\infty} \frac{\sin[(2n - 1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left|\frac{\sin[(2n - 1)\pi/2]}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p-series. Therefore, the series converges conditionally.

55. An alternating series is a series whose terms alternate in sign.

56. See Theorem 9.14.

57. $|S - S_N| = |R_N| \leq a_{N+1}$ (Theorem 9.15)

58. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.

59. (a) False. For example, let $a_n = \frac{(-1)^n}{n}$.

Then $\sum a_n = \sum \frac{(-1)^n}{n}$ converges

and $\sum(-a_n) = \sum \frac{(-1)^{n+1}}{n}$ converges.

But, $\sum |a_n| = \sum \frac{1}{n}$ diverges.

(b) True. For if $\sum |a_n|$ converged, then so would Σa_n by Theorem 9.16.

60. (b). The partial sums alternate above and below the horizontal line representing the sum.

61. True. $S_{100} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100}$

Because the next term $-\frac{1}{101}$ is negative, S_{100} is an overestimate of the sum.

62. False. Let

$$\sum a_n = \sum b_n = \sum \frac{(-1)^n}{\sqrt{n}}.$$

Then both converge by the Alternating Series Test. But,

$\sum a_n b_n = \sum \frac{1}{n}$, which diverges.

63. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If $p = 0$, then $\sum_{n=1}^{\infty} (-1)^n$ diverges.

If $p < 0$, then $\sum_{n=1}^{\infty} (-1)^n n^{-p}$ diverges.

If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$ and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for $p > 0$.

64. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+p}$

Assume that $n + p \neq 0$ so that $a_n = 1/(n+p)$ are defined for all n. For all p,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+p} = 0$$

$$a_{n+1} = \frac{1}{n+1+p} < \frac{1}{n+p} < a_n.$$

Therefore, the series converges for all p.

65. Because

$$\sum_{n=1}^{\infty} |a_n|$$

converges you have $\lim_{n\to\infty} |a_n| = 0$. So, there must exist an $N > 0$ such that $|a_N| < 1$ for all $n > N$ and it follows that $a_n^2 \le |a_n|$ for all $n > N$. So, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

66. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

67. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

68. (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for $-1 < x < 1$, because

$$\left|\frac{x^n}{n}\right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

(b) When $x = -1$, you have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, you have the divergent harmonic series $1/n$. Therefore,

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges conditionally for $x = -1$.

69. (a) No, the series does not satisfy $a_{n+1} \le a_n$ for all n. For example, $\frac{1}{9} < \frac{1}{8}$.

(b) Yes, the series converges.

$$\begin{aligned} S_{2n} &= \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} \\ &= \left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(\frac{1}{3} + \cdots + \frac{1}{3^n}\right) \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^n}\right) - \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^n}\right) \end{aligned}$$

As $n \to \infty$,

$$S_{2n} \to \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}.$$

70. (a) No, the series does not satisfy $a_{n+1} \le a_n$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and}$$

$$\frac{1}{8} < \frac{1}{\sqrt{3}}.$$

(b) No, the series diverges because $\sum \frac{1}{\sqrt{n}}$ diverges.

71. $\sum_{n=1}^{\infty}\frac{10}{n^{3/2}} = 10\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$,

convergent p-series

72. $\sum_{n=1}^{\infty}\frac{3}{n^2+5}$

converges by limit comparison to convergent p-series

$\sum \frac{1}{n^2}$.

73. Diverges by nth-Term Test

$\lim_{n\to\infty} a_n = \infty$

74. Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

75. Convergent geometric series

$\left(r = \frac{7}{8} < 1\right)$

76. Diverges by nth-Term Test

$\lim_{n\to\infty} a_n = \frac{3}{2}$

77. Convergent geometric series $\left(r = 1/\sqrt{e}\right)$ or Integral Test

78. Converges (conditionally) by Alternating Series Test

79. Converges (absolutely) by Alternating Series Test

80. Diverges by comparison to Divergent Harmonic Series:

$\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

81. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

82. $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

(i) $s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$

$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$

Adding: $s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = s_{3n}$

(ii) $\lim_{n\to\infty} s_n = s$ (In fact, $s = \ln 2$.)

$s \ne 0$ because $s > \frac{1}{2}$.

$S = \lim_{n\to\infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$

So, $S \ne s$.

Section 9.6 The Ratio and Root Tests

1. $\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = (n+1)(n)(n-1)$

2. $\frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$