Key

## BC Calculus – 9.5a & 9.5b - Derivatives and Integrals for Vector-Valued Functions

Vector basics:

- Vectors have magnitude (length) and direction.
- Vectors can be represented by directed line segments.
- Vectors are equal if they have the same direction and magnitude.
- Magnitude is designated by $\|v\|$
- Vectors have a horizontal and vertical component.
- Component form of a vector is $\langle x, y\rangle$

1. Find the component form and magnitude of the vector that has an initial point of $(1,2)$ and terminal point $(5,4)$.

$\langle x_2 - x_1, y_2 - y_1\rangle$ Component form: $\langle 5-1, 4-2\rangle = \langle 4, 2\rangle$

Magnitude: $\|r\| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$

**Vector-Valued Functions**: $r(t) = \langle f(t), g(t)\rangle$ where $f(t)$ and $g(t)$ are the component functions with the parameter $t$. *similar to parametric equations

### Differentiation of Vector-Valued Functions

If $r(t) = \langle f(t), g(t)\rangle$ then

$$r'(t) = \langle f'(t), g'(t)\rangle$$

**Properties of the derivative for vector-valued functions**

$$\frac{d}{dt}[c\cdot r(t)] = c\cdot r'(t) \qquad \frac{d}{dt}[r(t)\cdot s(t)] = r'(t)s(t) + r(t)s'(t)$$

$$\frac{d}{dt}[r(t) \pm s(t)] = r'(t) \pm s'(t) \qquad \frac{d}{dt}[r(s(t))] = r'(s(t))\cdot s'(t)$$

1. $r(t) = \langle 2t^2+4t+1,\ 3t^3-4t\rangle$ then $r'(t) =$

$$\boxed{r'(t) = \langle 4t+4, 9t^2-4\rangle}$$

2. $r(t) = \langle t^3+5,\ 2t\rangle$ find $\frac{d}{dt}r(2t) = r'(2t)\cdot 2$

Chain Rule

$r'(t) = \langle 3t^2, 2\rangle$

$r'(2t) = \langle 3(2t)^2, 2\rangle \cdot 2$

$r'(2t) = \langle 12t^2, 2\rangle \cdot 2 = \boxed{\langle 24t^2, 4\rangle}$

3. The path of a particle moving along a path in the $xy$-plane is given by the vector-valued function, $f(t) = \langle t^2, \sin t\rangle$. Find the slope of the path of the particle at $t = \frac{3\pi}{4}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{2t}\Bigg|_{t=3\pi/4} = \frac{\cos(3\pi/4)}{2(3\pi/4)} = \frac{-\sqrt{2}/2}{3\pi/2} \rightarrow \frac{-\sqrt{2}}{2}\cdot\frac{2}{3\pi} = \boxed{\frac{-\sqrt{2}}{3\pi}}$$

## 9.5b - Integrals for Vector-Valued Functions

**Integration of Vector-Valued Functions** *take the integral of horizontal and vertical components separately.

If $r(t) = \langle f(t), g(t) \rangle$ then

$$\int r(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt \right\rangle$$

1. Find $r(t)$ if $r'(t) = \langle 4e^{2t}, 2e^{t} \rangle$ and $r(0) = \langle 2, 0 \rangle$

$x = \int 4e^{2t}\,dt$

$u = 2t$, $\frac{du}{dt} = 2$, $dt = \frac{du}{2}$

$x = \int 4e^{u} \cdot \frac{du}{2}$

$x = 2e^{2t} + C_1$

$2 = 2e^{0} + C_1$

$0 = C_1$

$y = \int 2e^{t}\,dt$

$y = 2e^{t} + C_2$

$0 = 2e^{0} + C_2$

$-2 = C_2$

$\boxed{r(t) = \langle 2e^{2t}, 2e^{t} - 2 \rangle}$

2. Find $r(t)$ if $r'(t) = \langle \sec^2 t, \frac{1}{1+t^2} \rangle$

$x = \int \sec^2 t\,dt$

$x = \tan(t) + C_1$

$y = \int \frac{1}{1+t^2}\,dt$

$y = \arctan(t) + C_2$

$\boxed{r(t) = \langle \tan(t) + C_1, \tan^{-1}(t) + C_2 \rangle}$

3. $\int_{-1}^{1} \langle t^3, t^{\frac{1}{5}} \rangle\,dt$

$\int t^3\,dt \rightarrow \left.\frac{t^4}{4}\right]_{-1}^{1} = \frac{1}{4} - (1/4) = 0$

$\int t^{1/5}\,dt \rightarrow \frac{t^{6/5}}{6/5} \rightarrow \left.\frac{5}{6}t^{6/5}\right]_{-1}^{1} = \frac{5}{6}(1) - \frac{5}{6}(1) = 0$

$\boxed{\langle 0, 0 \rangle}$

$\boxed{f(t) = \langle -\frac{1}{2}e^{-t^2} + 1, e^{-t} - 2 \rangle}$

**For problems 1-6, find the vector-valued function $f(t)$ that satisfies the given initial conditions.**

1. $f(0) = \langle 2, 4 \rangle$, $f'(t) = \langle 2e^{t}, 3e^{3t} \rangle$

$x = \int 2e^{t} = 2e^{t} + C_1$

$2 = 2e^{0} + C_1$

$0 = C_1$

$y = \int 3e^{3t}\,dt$

$y = e^{3t} + C_2$

$4 = e^{0} + C_2$

$3 = C_2$

$\boxed{f(t) = \langle 2e^{t}, e^{3t} + 3 \rangle}$

2. $f(0) = \langle \frac{1}{2}, -1 \rangle$, $f'(t) = \langle te^{-t^2}, -e^{-t} \rangle$

$x = \int te^{-t^2}\,dt$

$u = -t^2$, $\frac{du}{dt} = -2t$, $dt = \frac{du}{-2t}$

$x = \int te^{u} \cdot \frac{du}{-2t}$

$x = -\frac{1}{2}e^{-t^2} + C_1$

$\frac{1}{2} = -\frac{1}{2}e^{0} + C_1$

$1 = C_1$

$y = \int -e^{-t}\,dt$

$y = e^{-t} + C_2$

$-1 = e^{0} + C_2$

$-2 = C_2$

9.5a **Derivatives of Vector-Valued Functions**

Calculus

**Practice**

**Each problem contains a vector-valued function. Find the given first or second derivative.**

1. $f(t) = \langle 4t^3 + 2t^2 + 7t,\ 4t^2 + 3t \rangle$, then $f'(t) =$

$\boxed{f'(t) = \langle 12t^2 + 4t + 7,\ 8t + 3 \rangle}$

2. $f(t) = \langle 3\sin 2t, 4\cos 3t \rangle$, then $f'\left(\frac{\pi}{6}\right) =$

$f'(t) = \langle 3\cos(2t)\cdot 2,\ -4\sin(3t)\cdot 3 \rangle$

$f'(\pi/6) = \langle 6\cos(2(\pi/6)),\ -12\sin(3\cdot\pi/6) \rangle$

$f'(\pi/6) = \langle 6\cdot \frac{1}{2},\ -12(1) \rangle = \boxed{\langle 3, -12 \rangle}$

3. $f(t) = \langle 3e^{2t},\ 5e^{4t} \rangle$, then $f''(t) =$

$f'(t) = \langle 3e^{2t}\cdot 2,\ 5e^{4t}\cdot 4 \rangle$

$f''(t) = \langle 6e^{2t}\cdot 2,\ 20e^{4t}\cdot 4 \rangle$

$\boxed{f''(t) = \langle 12e^{2t},\ 80e^{4t} \rangle}$

4. $f(t) = \langle t^{-2},\ (t+1)^{-1} \rangle$, then $f''(-2) =$

$f'(t) = \langle -2t^{-3},\ -1(t+1)^{-2} \rangle$

$f''(t) = \langle 6t^{-4},\ 2(t+1)^{-3} \rangle$

$f''(-2) = \langle \frac{6}{(-2)^4},\ \frac{2}{(-1)^3} \rangle$

$f''(-2) = \langle \frac{6}{16},\ -2 \rangle$

$\boxed{f''(-2) = \langle \frac{3}{8},\ -2 \rangle}$

5. $f(t) = \langle e^t + e^{-t},\ e^t - e^{-t} \rangle$, then $f'(t) =$

$\boxed{f'(t) = \langle e^t - e^{-t},\ e^t + e^{-t} \rangle}$

6. $f(t) = \langle 2\sin 4t, 2\cos 3t \rangle$, then $f'(t) =$

$f'(t) = \langle 2\cos(4t)\cdot 4,\ -2\sin(3t)\cdot 3 \rangle$

$\boxed{f'(t) = \langle 8\cos(4t),\ -6\sin(3t) \rangle}$

7. $f(t) = \langle t\sin t,\ t\cos t \rangle$, then $f'\left(\frac{\pi}{2}\right) =$

$f'(t) = \langle 1\sin t + t\cos t,\ 1\cos t - t\sin t \rangle$

$f'(\pi/2) = \langle \sin\pi/2 + 0,\ \cos(\pi/2) - \pi/2(1) \rangle$

$\boxed{f'(\pi/2) = \langle 1,\ -\pi/2 \rangle}$

8. $f(t) = \langle 3t^2 + 6t + 1,\ 4t^3 - 2t^2 + 6t \rangle$, then $f'(1) =$

$f'(t) = \langle 6t + 6,\ 12t^2 - 4t + 6 \rangle$

$f'(1) = \langle 12,\ 12 - 4 + 6 \rangle$

$\boxed{f'(1) = \langle 12, 14 \rangle}$

9. The path of a particle moving along a path in the $xy$-plane is given by the vector-valued function, $f(t) = \langle t^3 + 2t^2 + t,\ 2t^3 - 4t \rangle$. Find the slope of the path of the particle at $t = 3$.

$$f'(t) = \langle 3t^2+4t+1,\ 6t^2-4 \rangle$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left.\frac{6t^2-4}{3t^2+4t+1}\right|_{t=3} = \frac{54-4}{27+12+1} \rightarrow \frac{50}{40} = \boxed{\frac{5}{4}}$$

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10. The position of a particle moving in the $xy$-plane is defined by the vector-valued function, $f(t) = \langle t^3 - 6t^2,\ 2t^3 - 9t^2 - 24t \rangle$. For what value of $t \geq 0$ is the particle at rest?

*particle at rest when $x'(t) = 0$ and $y'(t) = 0$.

$x'(t) = 3t^2 - 12t$

$0 = 3t(t-4)$

$t = 0,\ \underline{t = 4}$

$y'(t) = 6t^2 - 18t - 24$

$0 = 6(t^2 - 3t - 4)$

$0 = 6(t-4)(t+1)$

$\underline{t = 4},\ t = -1$

$\boxed{t = 4}$

## 9.5a Derivatives of Vector-Valued Functions

**Test Prep**

11. **Calculator active.** The path of a particle moving along a path in the $xy$-plane is given by the vector-valued function $f$ and $f'$ is defined by $f'(t) = \langle t^{-1},\ 2ke^{kt} \rangle$ where $k$ is a positive constant. The line $y = 4x + 5$ is parallel to the line tangent to the path of the particle at the point where $t = 2$. What is the value of $k$?

slope

$\underline{m = 4}$

$$\frac{dy}{dx} = \left.\frac{2ke^{kt}}{t^{-1}}\right|_{t=2} = \frac{2ke^{2k}}{2^{-1}} = 4ke^{2k}$$

$\downarrow$

$4 = 4ke^{2k}$

$1 = ke^{2k}$

$ke^{2k} - 1 = 0$

$\boxed{k \approx 0.426}$

12. At time $t$, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the $xy$-plane is given by the vector-valued function, $f(t) = \langle t \sin t, \cos 2t \rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left.\frac{-\sin(2t)\cdot 2}{1\sin(t) + t\cdot\cos(t)}\right|_{t=\pi/2} = \frac{-2\sin(2\cdot\pi/2)}{\sin(\pi/2) + \pi/2\cos(\pi/2)} = \frac{0}{1} = \boxed{0}$$

# 9.5b Find $f(t)$

3. $f(0) = \langle 3, 1\rangle$, $f'(t) = \langle 6t^2, 4t\rangle$

$x = \int 6t^2 dt \qquad y = \int 4t\,dt$

$x = \frac{6t^3}{3} + C_1 \qquad y = \frac{4t^2}{2} + C_2$

$3 = 2(0) + C_1 \qquad 1 = 2(0)^2 + C_2$

$3 = C_1 \qquad 1 = C_2$

$$\boxed{f(t) = \langle 2t^3 + 3,\ 2t^2 + 1\rangle}$$

4. $f(0) = \langle -2, 5\rangle$, $f'(t) = \langle 2\cos t, -3\sin t\rangle$

$x = \int 2\cos(t)\,dt \qquad y = \int -3\sin(t)\,dt$

$x = 2\sin(t) + C_1 \qquad y = 3\cos(t) + C_2$

$-2 = 2\sin 0 + C_1 \qquad 5 = 3\cos(0) + C_2$

$-2 = C_1 \qquad 5 = 3 + C_2$

$2 = C_2$

$$\boxed{f(t) = \langle 2\sin(t) - 2,\ 3\cos(t) + 2\rangle}$$

5. $f'(0) = \langle 3, 0\rangle, f(0) = \langle 0, 3\rangle$,
$f''(t) = \langle 5\cos t, -2\sin t\rangle$

$x' = \int 5\cos(t)\,dt \qquad y' = \int -2\sin(t)\,dt$

$x' = 5\sin(t) + C \qquad y' = 2\cos(t) + C$

$3 = 5\sin(0) + C \qquad 0 = 2\cos(0) + C \qquad -2 = C$

$3 = C \qquad y = \int 2\cos(t) - 2\,dt$

$x' = 5\sin(t) + 3 \qquad y = 2\sin(t) - 2t + C$

$x = \int 5\sin(t) + 3\,dt \qquad 3 = 2\sin(0) - 2(0) + C \qquad 3 = C$

$x = -5\cos(t) + 3t + C \qquad y = 2\sin(t) - 2t + 3$

$0 = -5\cos(0) + 0 + C$

$5 = C$

$$\boxed{f(t) = \langle -5\cos(t) + 3t + 5,\ 2\sin(t) - 2t + 3\rangle}$$

6. $f'(0) = \langle 0, 2\rangle, f(0) = \langle 3, 0\rangle, f''(t) = \langle 4t^3, 3t^2\rangle$

$x' = \int 4t^3 dt \qquad y' = \int 3t^2 dt$

$x' = \frac{4t^4}{4} + C \qquad y' = \frac{3t^3}{3} + C$

$0 = C \qquad 2 = 0 + C$

$x = \int t^4 + 0\,dt \qquad y = \int t^3 + 2\,dt$

$x = \frac{t^5}{5} + C \qquad y = \frac{t^4}{4} + 2t + C$

$3 = 0 + C \qquad 0 = 0 + 0 + C \qquad 0 = C$

$x = \frac{1}{5}t^5 + 3$

$$\boxed{f(t) = \langle \tfrac{1}{5}t^5 + 3,\ \tfrac{1}{4}t^4 + 2t\rangle}$$

7. **Calculator active.** For $t \geq 0$, a particle is moving along a curve so that its position at time $t$ is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, 4)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+3}}{e^t}$ and $\frac{dy}{dt} = \cos^2 t$. Find the $x$-coordinate of the particles position at time $t = 5$.

*final position = initial position + displacement

$x(b) = x(a) + \int_a^b x'(t)\,dt$

$x(5) = x(1) + \int_1^5 \frac{\sqrt{t+3}}{e^t}\,dt$

$x(5) = 2 + 0.7988$

$$\boxed{x(5) = 2.7988}$$

8. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 8t^3 + 2t, 10t^4\rangle$. If $f(1) = \langle 3, 7\rangle$, what is $f(-1)$?

$x(b) = x(a) + \int_a^b x'(t)\,dt$

$x(-1) = x(1) + \int_1^{-1} 8t^3 + 2t\,dt$

$x(-1) = 3 + \left[\frac{8t^4}{4} + \frac{2t^2}{2}\right]_1^{-1} = 2(1)^4 + 1^2 - [2(1)^4 + 1]$

$x(-1) = 3 + 3 - 3 = 3$

$y(b) = y(a) + \int_a^b y'(t)\,dt$

$y(-1) = y(1) + \int_1^{-1} 10t^4\,dt$

$y(-1) = 7 + \left[\frac{10t^5}{5}\right]_1^{-1} = 2(-1)^5 - 2(1)^5$

$y(-1) = 7 - 2 - 2 = 3$

$$\boxed{f(-1) = \langle 3, 3\rangle}$$

9. **Calculator active.** At time $t \geq 0$, a particle moving in the $xy$-plane has velocity vector given by $v(t) = \langle 3t^2, 3\rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

$x(b) = x(a) + \int_a^b v(t)\,dt$

$x(2) = x(0) + \int_0^2 3t^2\,dt$

$x(2) = 1 + \frac{3t^3}{3}\Big]_0^2 = 8$

$x(2) = 9$

$y(2) = y(0) + \int_0^2 3\,dt$

$y(2) = 2 + 3t\Big]_0^2$

$= 2 + 6 - 0 = 8$

$y(2) = 8$

distance from origin: $\sqrt{9^2 + 8^2} \approx \boxed{12.0415}$

10. **Calculator active.** At time $t \geq 0$, a particle moving in the $xy$-plane has velocity vector given by $v(t) = \langle 2, \frac{\cos t}{e^t}\rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

$x(3) = x(0) + \int_0^3 2\,dt \rightarrow 1 + 2t\Big]_0^3 \rightarrow 1 + 6 - 0 = 7$

$y(3) = y(0) + \int_0^3 \frac{\cos(t)}{e^t}\,dt \rightarrow 2 + 0.52815 = 2.52815$

Distance from origin: $\sqrt{7^2 + (2.52815)^2}$

$\boxed{\approx 7.4425}$

## $\boxed{9.5b}$ Integrating Vector-Valued Functions

**Test Prep**

11. **Calculator active.** A remote controlled car travels on a flat surface. The car starts at the point with coordinates $(7, 6)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position change at rates given by $x'(t) = -10\sin t^2$ and $y'(t) = 9\cos(2 + \sqrt{t})$, where $x(t)$ and $y(t)$ are measured in feet and $t$ is measured in minutes. Find the $y$-coordinate of the position of the car at time $t = 1$.

$y(1) = y(0) + \int_0^1 y'(t)\,dt$

$y(1) = 6 + \int_0^1 9\cos(2+\sqrt{t})\,dt$

$y(1) = 6 + -7.78906$

$\boxed{y(1) \approx -1.789}$

12. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 2 + 20t - 4t^3,\ 6t^2 + 2t\rangle$. If $f(1) = \langle 5, -3\rangle$, what is $f(-1)$?

$x(-1) = x(1) + \int_1^{-1} 2 + 20t - 4t^3\,dt$

$x(-1) = 5 + 2t + \frac{20t^2}{2} - \frac{4t^4}{4}\Big]_1^{-1}$

$5 + [-2 + 10 - 1] - [2 + 10 - 1]$

$= 5 + 7 - 11 = 1$

$y(-1) = y(1) + \int_1^{-1} 6t^2 + 2t\,dt$

$y(-1) = -3 + \left[\frac{6t^3}{3} + \frac{2t^2}{2}\right]\Big]_1^{-1}$

$= -3 + (-2+1) - (2+1) = -3 - 1 - 3 = -7$

$\boxed{f(-1) = \langle 1, -7\rangle}$