AP Calculus BC 9.5a Notes Alternating Series Test (AST)

All of the tests that we've used so far have dealt with only positive terms. This section, 9.5, and the following section will have tests to determine convergence of series with both positive and negative consecutive terms. The simplest such series is an **alternating series**. An alternating series is a series whose terms alternate signs.

For an example, consider the geometric series: $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$

This series is an alternating geometric series with r = $-\frac{1}{2}$.

Alternating Series Test

Let $a_n > 0$. The alternating series: $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ will converge if

the following two conditions are met. 1) $\lim_{n\to\infty} a_n = 0$ and 2) $a_{n+1} \le a_n$ for all n

Note 1: For an Alternating Series, the series must converge if its terms consistently shrink in size and approach zero.

Note 2: AST does NOT show divergence if the conditions for convergence are not met. (This is only a 1 way test for convergence. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the **nth term test**, NOT by AST.

Example 1: Determine whether the following series converge or diverge

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2n-1}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n} n}{\ln(2n)}$

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$

(e) $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(n-5)^2+1}$

(f) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

g) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+1)}{n}$

h) $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$

i) $\frac{2}{1}-\frac{1}{1}+\frac{2}{2}-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}+\frac{2}{4}-\frac{1}{4}+\ldots$

AP Calculus BC 9.5a Notes Alternating Series Test (AST) Key

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For an example, consider the geometric series: $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\underbrace{(-1)^n}_{\boxed{\text{alternator}}}\left(\frac{1}{2}\right)^n = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$

↳ contributes to the sign changes
* also $(-1)^{n+1}$, $(-1)^{n-1}$

This series is an alternating geometric series with r = $-\frac{1}{2}$.

Alternating Series Test (AST)

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Example 1: Determine whether the following series converge or diverge

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2n-1}$ diverges by nth term test. $\lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2} \neq 0$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{\ln(2n)}$ By nth term test $\lim_{n\to\infty}\frac{n}{\ln(2n)} \neq 0$, so series diverges.

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n} = (-1)^n \cdot \frac{1}{n}$

* Disguised alternating series.

since $\lim_{n\to\infty}\frac{1}{n} = 0$, by AST, series converges.

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!} = (-1)^{n-1}\left(\frac{1}{n!}\right)$

since $\lim_{n\to\infty}\frac{1}{n!} = 0$, by AST, series converges.

(e) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-5)^2+1}$

since $\lim_{n\to\infty}\frac{1}{(n-5)^2+1} = 0$, By AST, series converges.

(f) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = (-1)^{n+1}\left(\frac{1}{n}\right)$

$\lim_{n\to\infty}\frac{1}{n} = 0$, so by AST, series converges.

g) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

$\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0,$

series diverges by n^{th} term test.

h) $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

i) $\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \dots$

$2 - 1 + 1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

a) $\lim_{n\to\infty} a_n = 0$ ✓

b) $a_{n+1} \leq a_n$ for all n (false)

so AST does not apply for above series.