AP Calculus BC 9.5b Notes Alternating Series Remainder and Convergence

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

Alternating Series Remainder:

Suppose an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing. If the series has a sum S, then $|R_n| = |S - S_n| \le a_{n+1}$, where S_n is the nth partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will have an absolute value no greater than the first term left off, a_{n+1}. This means $S \in [S_n - R_n, S_n + R_n]$

Example 2: Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error. Use your results to find an interval in which S must lie.

Example 3: Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \ldots$$

Example 4: Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001

Example 5: Determine the number of terms required to have an error less than 0.001

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}$$

For series that have positive and negative terms but are not alternating (i.e. -,-,+,+,-,-,+,+), the following test can show whether the original series converges or diverges

The converse of the absolute convergence theorem is not true. For instance, the alternating harmonic series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges by the Alternating Series Test. However, the absolute value of this series (the normal harmonic) and is divergent by the p-series test. If the absolute value of the series is divergent and the original series is convergent, we call this series **conditionally convergent**.

Definitions of Absolute and Conditional Convergence

1) $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.
2) $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 1: Determine Convergence or Divergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$$

Example 2: Determine Convergence or Divergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$$

AP Calculus BC 9.5b Notes Alternating Remainder and Convergence

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

Alternating Series Remainder: *a convergent series*

Suppose an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing. If the series has a sum S, then $|R_n| = |S - S_n| \le a_{n+1}$, where S_n is the nth partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will have an absolute value no greater than the first term left off, a_{n+1}. This means $S \in [S_n - R_n, S_n + R_n]$

Example 2: Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error. Use your results to find an interval in which S must lie.

*This is a convergent AST.

sum(seq((-1)^(x-1)/x!, x, 1, 6)) = 0.6319 error is $\left|\frac{1}{7!}\right| \approx 1.98\times10^{-4}$

$S \in \left[0.6319 - \frac{1}{7!}, 0.6319 + \frac{1}{7!}\right]$

Example 3: Approximate the sum of the following series by its first six terms. (7th term is first unused, neglected term)

$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \ldots$

Sum of series is

$0.63194 - 0.002 \le S \le 0.63194 + 0.002$

$\boxed{0.63174 \le S \le 0.63214}$

Remainder $R_6 \le |a_7| = \frac{1}{7!} = \frac{1}{5040} = 0.002$

$S_6 = 0.63194$

first unused term in the sum

Example 4: Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001

$= \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4}$

$\frac{1}{6^4} \approx 7.716\times10^{-4} = 0.000716$

so $S_5 = 0.947$

Example 5: Determine the number of terms required to have an error less than 0.001

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$

$\frac{1}{2n-1} = 0.001$

$1 = 0.001(2n-1)$

$1 = 0.002n - 0.001$

$1.001 < 0.002n$

$500.5 < n$

$\boxed{\text{501 terms required}}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}$

$\frac{1}{(n+1)!} < 0.001$

If $n=6$, $\frac{1}{7!} = 0.000198$

$\boxed{\text{Number of terms required is } n > 5}$

9.5b (continued)

For series that have positive and negative terms but are not alternating (i.e. -,-,+,+,-,-,+,+), the following test can show whether the original series converges or diverges

The converse of the absolute convergence theorem is not true. For instance, the alternating harmonic series: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ converges by the Alternating Series Test. However, the absolute value of this series (the normal harmonic) and is divergent by the p-series test. If the absolute value of the series is divergent and the original series is convergent, we call this series **conditionally convergent**.

If a series converges when all terms are made positive, then the original series will converge.

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the series converges on the condition that it needs the negative values from alternator in order to make it convergent.

Example of conditionally convergent series

$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)$

Example 6: Determine Convergence or Divergence

$S=\sum_{n=1}^{\infty}\frac{(-1)^{n(n+1)/2}}{3^n}$ As is, series converges by AST, since $\lim_{n\to\infty}\frac{1}{3^n}=0$.

since $\lim_{n\to\infty}\left|\frac{(-1)^{n(n+1)/2}}{3^n}\right|$ converges by GST, this series converges absolutely.

Example 7: Determine Convergence or Divergence

$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n^2}}$ As is, this series converges by AST since $\lim_{n\to\infty}\frac{1}{n^{2/3}}=0$

However $\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ diverges by p-series test, $p=\frac{2}{3}<1$

So series converges conditionally.