Key

BC Calculus – 9.6 Motion Using Parametrics and Vectors Notes

Position: $r(t) = \langle x(t), y(t) \rangle$

Velocity: $v(t) = r'(t) = \langle x'(t), y'(t) \rangle$

Acceleration: $a(t) = r''(t) = \langle x''(t), y''(t) \rangle$

(speed is not a vector)

Speed: $\|v(t)\| = \|r'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

speed is a magnitude of velocity vector

1. Find the velocity vector, speed, and acceleration vector for the particle that moves in the xy-plane described by $r(t) = \langle 5\sin\frac{t}{5}, 5\cos\frac{t}{5} \rangle$

$v(t) = \langle 5\cos(t/5)\cdot\frac{1}{5}, -5\sin(t/5)\cdot\frac{1}{5} \rangle = \langle \cos(t/5), -\sin(t/5) \rangle$

$a(t) = \langle -\sin(t/5)\cdot\frac{1}{5}, -\cos(t/5)\cdot\frac{1}{5} \rangle = \langle -\frac{1}{5}\sin(t/5), -\frac{1}{5}\cos(t/5) \rangle$

speed $= \sqrt{\cos^2(t/5) + \sin^2(t/5)} = \sqrt{1} = \boxed{1}$

Quick review: When does a particle's speed increase or decrease?

Speeding up

Velocity & Acceleration have the same sign

Slowing down

Velocity & Acceleration have different signs

position function

2. If $r(t) = \langle 2t^3 + t, t^2 \rangle$, find velocity and acceleration at time t.

$v(t) = \langle 6t^2 + 1, 2t \rangle$

$a(t) = \langle 12t, 2 \rangle$

3. Find the speed at time $t = 2$ if $r(t) = \langle 3t, e^{-t^2} \rangle$

$v(t) = \langle 3, e^{-t^2}\cdot -2t \rangle$

$v(2) = \langle 3, -4e^{-4} \rangle$

speed $= \sqrt{(3)^2 + \left(\frac{4}{e^4}\right)^2} \approx \boxed{3.00089}$

Total Distance Traveled by a Partice on $[a, b]$.

$$\int_a^b \|v(t)\|\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$v(t) = \langle x'(t), y'(t) \rangle$

4. Given the velocity vector of the particle $v(t) = \langle 2t+1,\ 5\rangle$ and the position of the particle at time $t = 0$ is $(1, 2)$, find the position when $t = 3$. What is the total distance traveled on the interval $0 \le t \le 3$?

$x(b) = x(a) + \int_a^b v(t)\,dt$

*Recall: Final position = Initial position + displacement

$x(3) = x(0) + \int_0^3 2t+1\,dt$

$x(3) = 1 + \frac{2t^2}{2} + t\Big]_0^3 \to 3^2 + 3$

$x(3) = 1 + 12 = 13$

$y(3) = y(0) + \int_0^3 5\,dt$

$2 + 5t\Big]_0^3 = 15$

$y(3) = 2 + 15 = 17$

$\boxed{P(3) = \langle 13, 17\rangle}$

b) Total Distance:

$\int_0^3 \sqrt{(2t+1)^2 + 5^2}\,dt$

$\approx \boxed{19.649}$

5. A particle moving along a curve so that its velocity for time $t \ge 0$ is given by $v(t) = \langle 2e^{-\frac{t}{4}}, \frac{t-4}{t+5}\rangle$.

a. For what values of t is the particle moving to the right?

*the horizontal direction of motion (left or right) is determined by the x component of the velocity vector.

Since $2e^{-t/4}$ is always greater than zero, particle is moving right when $t \ge 0$.

b. For what values of t is the particle moving up?

*vertical direction of motion (up, down) determined by y-component of $v(t)$.

Moves up when $y'(t) > 0$

Find when $\frac{t-4}{t+5} > 0$

$t - 4 > 0$

$t > 4$

$\boxed{\text{particle is moving up when } t > 4}$

Practice Problems:

For each problem, a particle moves in the xy-plane where the coordinates are defined at any time t by the position function given in parametric or vector form.

1. $x(t) = 4t^2$ and $y(t) = 2t - 1$. Find the velocity vector at time $t = 1$.

$x'(t) = 8t$ $\quad y'(t) = 2$

$x'(1) = 8$ $\quad y'(1) = 2$

$\boxed{v'(1) = \langle 8, 2\rangle}$

2. $x(t) = e^{-t}$ and $y(t) = e^t$. Find the acceleration vector at time $t = 1$.

$x'(t) = e^{-t}(-1)$ $\quad y'(t) = e^t$

$x''(t) = -e^{-t}(-1)$ $\quad y''(t) = e^t$

$a(t) = \langle \frac{1}{e^t}, e^t\rangle$

$\boxed{a(1) = \langle \frac{1}{e}, e^1\rangle}$

3. $(x(t), y(t)) = (6 - 2t, t^2 + 3)$. In which direction is the particle moving as it passes through the point $(4, 4)$?

$x(t) = 6 - 2t$
$4 = 6 - 2t$
$2t = 2$
$t = 1$

$y(t) = t^2 + 3$
$4 = t^2 + 3$
$1 = t^2 \quad t = 1, t = -1$
$t = 1$

$v(t) = \langle -2, 2t \rangle$

$v(1) = \langle -2, 2 \rangle$

Particle is moving left and up

4. A position vector is $r(t) = \langle \frac{2}{t}, e^{4t} \rangle$ for time $t > 0$. What is the velocity vector at time $t = 1$?

$r(t) = \langle 2t^{-1}, e^{4t} \rangle$

$r'(t) = \langle -2t^{-2}, e^{4t}(4) \rangle$

$r'(1) = \langle -2(1)^{-2}, 4e^{4(1)} \rangle$

$\boxed{r'(1) = \langle -2, 4e^4 \rangle}$

5. $r(t) = \langle \ln(t^2 + 1), 3t^2 \rangle$ for $t > 0$. Find the velocity vector at time $t = 2$.

$r'(t) = \langle \frac{2t}{t^2+1}, 6t \rangle$

$r'(2) = \langle \frac{2(2)}{2^2+1}, 6(2) \rangle$

$\boxed{v(2) = \langle \frac{4}{5}, 12 \rangle}$

6. $x(t) = 2\sin\frac{t}{2}$ and $y(t) = 2\cos\frac{t}{2}$ for time $t > 0$. Find the speed of the particle.

$x'(t) = 2\cos(t/2) \cdot \frac{1}{2} \qquad y'(t) = -2\sin(t/2) \cdot \frac{1}{2}$

speed $= \sqrt{\cos^2(t/2) + [-\sin(t/2)]^2}$

$= \sqrt{1} = \boxed{1}$

7. **Calculator active.** $x(t) = t^2 + 1$ and $y(t) = \frac{4}{3}t^3$ for time $t \geq 0$. Find the total distance traveled from $t = 0$ to $t = 3$.

$x'(t) = 2t \qquad y'(t) = \frac{4}{3} \cdot 3t^2$

distance $= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

8. $p(t) = \langle \cos 2t, 2\sin t \rangle$. Find the velocity vector $v(t)$.

$v(t) = \langle -\sin(2t) \cdot 2, 2\cos(t) \rangle$

$\boxed{v(t) = \langle -2\sin(2t), 2\cos(t) \rangle}$

9. **Calculator active.** The velocity vector of a particle moving in the xy-plane has components given by $\frac{dx}{dt} = \cos t^2$ and $\frac{dy}{dt} = e^{t-2}$. At time $t = 3$, the position of the particle is $(1, 2)$. What is the y-coordinate of the position vector at time $t = 2$?

$y(2) = y(3) + \int_3^2 y'(t)\,dt$

$y(2) = 2 + \int_3^2 e^{t-2}\,dt$

$y(2) = 2 + -1.71828 \approx \boxed{0.2817}$

10. At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle t^3, 4t\rangle$. What is the acceleration vector when $t = 2$?

$a(t) = \langle 3t^2, 4\rangle$

$\boxed{a(2) = \langle 12, 4\rangle}$

11. The acceleration vector of a particle moving in the xy-plane is given by $a(t) = \langle 2, 3\rangle$. When $t = 0$ the velocity vector is $\langle 3, 1\rangle$ and the position vector is $\langle 1, 5\rangle$. Find the position when time $t = 2$.

$v(t) = \langle 2t + C_1, 3t + C_2\rangle$

$v(0) = \langle 3, 1\rangle$

$3 = 2(0) + C_1$

$3 = C_1$

$1 = 3t + C_2$

$1 = 3(0) + C_2$

$1 = C_2$

$v(t) = \langle 2t+3, 3t+1\rangle$

$p(0) = \langle 1, 5\rangle$

$p(t) = \langle \frac{2t^2}{2} + 3t + C_1, \frac{3t^2}{2} + t + C_2\rangle$

$1 = 0^2 + 3(0) + C_1$

$5 = \frac{3}{2}(0)^2 + 0 + C_2$

$p(t) = \langle t^2 + 3t + 1, \frac{3}{2}t^2 + t + 5\rangle$

$p(2) = \langle 2^2 + 3(2) + 1, \frac{3}{2}(2)^2 + 2 + 5\rangle$

$\boxed{p(2) = \langle 11, 13\rangle}$

12. A particle moves on the curve $y = 2x$ so that the x-component has velocity $x'(t) = 3t^2 + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(2, 4)$. At what point is the particle when $t = 1$? [This one is tricky!]

$x(t) = \frac{3t^3}{3} + t + C$

$2 = 0 + 0 + C$ $\quad$ $C = 2$

$x(t) = t^3 + t + 2$

position vector

$\langle t^3 + t + 2, 2x\rangle$

$\langle t^3 + t + 2, 2(t^3 + t + 2)\rangle$

$\langle t^3 + t + 2, 2t^3 + 2t + 4\rangle$

$\boxed{\text{at } t = 1,\ p(1) = \langle 4, 8\rangle}$

For problems 13-15: At time t, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the xy-plane is given by parametric equations $x(t) = \cos 2t$ and $y(t) = \sin 2t$.

13. Find the speed of the particle when $t = 1$.

$\text{speed} = \sqrt{[x'(1)]^2 + [y'(1)]^2}$

$x'(t) = -\sin(2t) \cdot 2$ $\quad$ $y'(t) = \cos(2t) \cdot 2$

$\sqrt{[2\sin^2(2t)] + [2\cos^2(2t)]} = \sqrt{4(\sin^2(2t) + \cos^2(2t))}$

$= \sqrt{4} = \boxed{2}$

14. Find the acceleration vector at time $t = \frac{\pi}{4}$.

$x''(t) = -2\cos(2t) \cdot 2$ $\quad$ $y''(t) = -2\sin(2t) \cdot 2$

$x''(\pi/4) = -4\cos(2\pi/4)$ $\quad$ $y''(\pi/4) = -4\sin(2\pi/4)$

$\boxed{a(\pi/4) = \langle 0, -4\rangle}$

15. Find the distance traveled from $t = 0$ to $t = 3$.

$\text{distance} = \int_0^3 \sqrt{[2\sin(2t)]^2 + [2\cos(2t)]^2}\,dt$

$= \int_0^3 \sqrt{4}\,dt$

$2t\Big]_0^3 = 2(3) - 2(0)$

$= \boxed{6}$

16. **Calculator active.** A remote-controlled car moves along a flat surface over the time interval $0 \le t \le 30$ seconds. The position of the remote-controlled car at time t is given by the parametric equations $x(t) = 2t + \sin t$ and $y(t) = 2\cos(t - \sin t)$, where $x(t)$ and $y(t)$ are measured in feet. The derivatives of these functions are given by $x'(t) = 2 + \cos t$ and $y'(t) = -2\sin(t - \sin t)(1 - \cos t)$.

    a. Write the equation for the line tangent to the path of the remote-controlled car at time $t = 3$ seconds.

$x(3) = 2(3) + \sin(3) \approx 6.141$

$y(3) = 2\cos(3 - \sin 3) \approx -1.9206$

$\frac{dy}{dx} = \left.\frac{-2\sin(t-\sin t)(1-\cos t)}{2+\cos(t)}\right|_{t=3} \approx -1.099$

point: $(6.141, -1.9206)$

slope: $m \approx -1.099$

$\boxed{y + 1.921 = -1.099(x - 6.141)}$

    b. Find the speed of the remote-controlled car at time $t = 15$ seconds.

speed $= \sqrt{[x'(15)]^2 + [y'(15)]^2}$

$\approx \boxed{3.6569 \text{ feet/second}}$

    c. Find the acceleration vector of the remote-controlled car at the time when the car is at the point with x-coordinate 40.

$x = 40$

$x(t) = 2t + \sin(t)$

$2t + \sin(t) = 40$

$\underbrace{2t + \sin(t) - 40}_{\text{Graph this}} = 0$

Graph this and look for x-intercept

$t \approx 19.6434$

$x''(19.6434) \approx -0.713$

$y''(19.6434) \approx -0.293$

use calculator nderiv (math 8) on $x'(t)$ and $y'(t)$

$\boxed{\langle -0.713, -0.293 \rangle}$