AP Calculus BC 9.6 Notes Ratio Test and Root Test

-Use the ratio test to determine convergence or divergence

The **ratio test** is a test that determines whether a function converges absolutely.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.

3) The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

The Ratio Test is particularly useful for series that converge rapidly (i.e. factorials or exponentials).

Examples: Determine Convergence or Divergence

1) $\sum_{n=0}^{\infty}\frac{2^n}{n!}$

2) $\sum_{n=0}^{\infty}\frac{n^2 2^{n+1}}{3^n}$

3) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

4) $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

5) $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

9.6 Notes Continued: Root Test

The final test to determine convergence or divergence is the root test. The root test is especially well suited to solve series involving n^{th} powers.

Root Test

Let $\sum a_n$ be a series.

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3) The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Example: Using the Root Test

6) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

7) $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^n$

8) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

9) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

10) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

AP Calculus BC **9.6 Notes** **Ratio Test and Root Test** Key

-Use the ratio test to determine convergence or divergence

The **ratio test** is a test that determines whether a function converges absolutely.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms. → divide the subsequent term by the previous term.

* similar to LCT, except we are using the same series.

* good test for exponential, factorial series. (functions that grow rapidly)

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.

3) The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

The Ratio Test is particularly useful for series that converge rapidly (i.e. factorials or exponentials).

Examples: Determine Convergence or Divergence

1) $\sum_{n=0}^{\infty}\frac{2^n}{n!}$

n^{th} term test → inconclusive.

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$

replace n with "$n+1$" to arrive at $(n+1)^{st}$ term

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1,$$

series converge absolutely. (by Ratio Test)

2) $\sum_{n=0}^{\infty}\frac{n^2 2^{n+1}}{3^n}$

n^{th} term test inconclusive

$$\lim_{n\to\infty}\left|\frac{(n+1)^2\cdot 2^{n+2}}{3^{n+1}}\cdot\frac{3^n}{n^2\cdot 2^{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2\cdot 2}{n^2\ \ 3} = \frac{2}{3} < 1,$$

series converge (by Ratio Test)

3) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right|$$

$$= \frac{(n+1)^n\cancel{(n+1)}\cdot\cancel{n!}}{\cancel{(n+1)}\cdot\cancel{n!}\cdot n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

so series diverge by Ratio Test.

4) $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}}\right| = \left(\frac{\sqrt{n+1}}{\sqrt{n}}\right)\left(\frac{n+1}{n+2}\right)$$

Ratio Test Inconclusive

By AST, since $\lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = 0$,

so series converges.

5) $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

$\boxed{\text{Diverges}}$ by n^{th} term test, $b^n < b!$

9.6 Notes Continued: Root Test

The final test to determine convergence or divergence is the root test. The root test is especially well suited to solve series involving n^{th} powers. *Resembles Ratio Test in some ways.

*Good for when the entire rule of sequence can be written as a power of n.

Root Test

Let $\sum a_n$ be a series.

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3) The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Example: Using the Root Test

6) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

n^{th} term test inconclusive

$$\sum_{n=1}^{\infty} \sqrt[n]{\left(\frac{e^2}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{e^2}{n} = 0 < 1,$$

so series converge by the root test.

7) $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{-3n}{2n+1}\right|^n} = \lim_{n\to\infty} \left|\frac{-3n}{2n+1}\right| = \frac{3}{2} > 1,$$

So series diverge by root test.

8) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{2}$$

let $y = \sqrt[n]{n} = n^{1/n}$

$\ln y = \ln n^{1/n}$

$\ln y = \left(\frac{1}{n}\right)\ln n$

$\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$\ln y = 0$

$e^0 = y$

$1 = y$

$y = \frac{1}{2} < 1,$

converges by root test.

9) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

$$\sqrt[n]{\left(\frac{n!}{n^2}\right)^n}$$

$$\lim_{n\to\infty} \left(\frac{n!}{n^2}\right) = \infty,$$

diverges by root test.

10) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

$$\sqrt[n]{\left(\frac{3n+4}{2n}\right)^n}$$

$$\lim_{n\to\infty} \frac{3n+4}{2n} = \frac{3}{2} > 1,$$

diverges by root test.