9.6 Ratio Test and Root Test   p.645 #13-71 odds

*Ratio Test: 1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

*useful for factorials, exponentials.

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.

3) Ratio test inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

13) $\sum_{n=0}^{\infty} \frac{n!}{3^n}$ $\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}}\right| = \frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!} = \lim_{n\to\infty}\frac{n+1}{3} = \boxed{\infty}$ series diverge

15) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$ $\lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{3}{4}\right)^{n+1}}{n\left(\frac{3}{4}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{3}{4}\right| = \frac{3}{4} < 1$ series converges

17) $\sum_{n=1}^{\infty}\frac{n}{2^n}$ $\lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{2^n}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{1}{2}\right| = \frac{1}{2} < 1$

converges by Ratio Test.

27) $\sum_{n=0}^{\infty}\frac{3^n}{(n+1)^n}$ $\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}}{(n+2)^{n+1}}}{\frac{3^n}{(n+1)^n}}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{3^n}\right| = \lim_{n\to\infty}\frac{3}{1}\cdot\left(\frac{n+1}{n+2}\right)^n\cdot\frac{1}{n+2}$

$= 0\cdot\left(\frac{n+1}{n+2}\right)^n = 0$ converges by Ratio Test

$0\cdot\left(\frac{1}{e}\right) = 0$

31) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cdot n!}{1\cdot 3\cdot 5\cdots(2n+1)}$

$\lim_{n\to\infty}\left|\frac{(n+1)!}{\cancel{(2n+1)}(2n+3)}\cdot\frac{\cancel{2n+1}}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{(2n+3)} = \frac{1}{2} < 1$

Converges by Ratio Test

$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{n!}}{\cancel{n!}}$

Root Test:

*well suited for series involving n^{th} powers

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$
3) Root test inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

37) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$ series converges by Root Test.

43) $\sum_{n=1}^{\infty} (2\sqrt[n]{n}+1)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{(2\sqrt[n]{n}+1)^n} = \lim_{n\to\infty}(2\sqrt[n]{n}+1)$

$y = \lim_{n\to\infty} \sqrt[n]{n}$ $\quad \ln y = \ln(n)^{1/n}$ $\quad \ln y = \frac{\ln n}{n} \to \lim_{n\to\infty} \frac{\ln n}{n} \xrightarrow{L'H} \frac{\frac{1}{n}}{1} = 0$ $\quad \ln y = 0$ $\quad e^0 = y$

$y = 1$

$= \lim_{n\to\infty} 2(1)+1 = \boxed{3}$ By Root Test, series diverges.

45) $\sum_{n=1}^{\infty} \frac{n}{4^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{n}{4^n}} = \frac{n^{1/n}}{4}$

let $y = \lim_{n\to\infty} n^{1/n}$ $\quad \ln y = \frac{\ln n}{n}$ $\quad \lim_{n\to\infty} \frac{\ln n}{n} \xrightarrow{L'H} \frac{\frac{1}{n}}{1} = 0$ $\quad \ln y = 0$ $\quad e^0 = y$

$\lim_{n\to\infty} \frac{n^{1/n}}{4} = \boxed{\frac{1}{4}} < 1$ Series diverge by Root Test. $\quad y = 1$

47) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{n^2} = 0 - 0 = 0 < 1$ Converges by Root Test

49) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{n^{1/n}}{\ln n} = \frac{1}{\infty} = 0 < 1$ Converges by Root Test.

51) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 5}{n}$ a) $\lim_{n\to\infty} \frac{5}{n} = 0$ b) $\frac{5}{n+1} < \frac{5}{n}$, so $a_{n+1} < a_n$ Converges by AST (conditional convergence)

53) $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = \frac{3}{n^{3/2}}$ p-series test: $p = 3/2 > 1$ converges by p-series test.

55) $\sum_{n=1}^{\infty} \frac{2n}{n+1}$ $\lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$ Diverges by n^{th} term test.

57) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^{n-2}}{2^n} = (-1)^n \cdot \left(\frac{3}{2}\right)^n \cdot \frac{1}{3^2} = \sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^n \cdot \frac{1}{9}$ $|r| = \frac{3}{2} > 1$ Diverges by geometric series test.

59) $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$ $\lim_{n\to\infty} \left| \frac{\frac{10(n+1)+3}{(n+1)\cdot 2^{n+1}}}{\frac{10n+3}{n\cdot 2^n}} \right| = \lim_{n\to\infty} \left| \frac{10n+13}{(n+1)2^{n+1}} \cdot \frac{n\cdot 2^n}{10n+3} \right|$

$a_n = \frac{10n+3}{n\cdot 2^n}$ $\lim_{n\to\infty} \left| \frac{\frac{10n+3}{n\cdot 2^n}}{\frac{1}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{10n+3}{n\cdot \cancel{2^n}} \cdot \frac{\cancel{2^n}}{1} \right| = \boxed{10}$

$b_n = \frac{1}{2^n}$

Since $\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right|$ converges, then series converge by Limit Comparison Test

61) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ $\left| \frac{\cos n}{2^n} \right| < \frac{1}{2^n}$

series converge by Direct Comparison Test with convergent geometric series.

63) $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$ $\lim_{n\to\infty} \left| \frac{\frac{(n+1)7^{n+1}}{(n+1)!}}{\frac{n7^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n\cdot 7^n} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{n+1} \cdot \frac{\cancel{7^n}\cdot 7}{n \cdot \cancel{7^n}} \right| = 0 < 1$

converges by Ratio Test.

65) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^{n-1}}{n!}$ $\lim_{n\to\infty} \left| \frac{\frac{3^n}{(n+1)!}}{\frac{3^{n-1}}{n!}} \right| = \lim_{n\to\infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{3}{n+1} \right| = 0 < 1$

Converges by Ratio Test
(Absolute convergence)

67) $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdot (2n+1)}$ $\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(2n+1)(2n+3)}}{\frac{3^n}{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(2n+1)(2n+3)} \cdot \frac{(2n+1)}{3^n} \right| = \lim_{n\to\infty} \left| \frac{3}{2n+3} \right| = 0 < 1$

Converges by Ratio Test
(Absolute)

69) a) $\sum_{n=1}^{\infty} \frac{n \cdot 5^n}{n!} = \frac{1 \cdot 5}{1} + \frac{2 \cdot 5^2}{2!} + \frac{3 \cdot 5^3}{3!} + \dots$

b) $\sum_{n=0}^{\infty} \frac{n \cdot 5^n}{(n+1)!} = \frac{0 \cdot 5}{1} + \frac{1 \cdot 5^1}{2} + \frac{2 \cdot 5^2}{3!} + \frac{3 \cdot 5^3}{4!}$

c) $\sum_{n=0}^{\infty} \frac{(n+1) \cdot 5^{n+1}}{(n+1)!}$ $\frac{0 \cdot 5^1}{1} + \frac{2 \cdot 5^2}{2!} + \frac{3 \cdot 5^3}{3!}$

(a) and (c) are the same series

71) a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ $\frac{1}{1!} + \frac{-1}{3!} + \frac{1}{5!}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$ $\frac{+1}{\cancel{0}!} + \frac{-1}{3!} + \frac{1}{5!} + \dots$

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$ $\frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!}$

(a) and (b) are the same series.