70. (a) No, the series does not satisfy $a_{n+1} \le a_n$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and}$$

$$\frac{1}{8} < \frac{1}{\sqrt{3}}.$$

(b) No, the series diverges because $\sum \frac{1}{\sqrt{n}}$ diverges.

71. $\displaystyle\sum_{n=1}^{\infty}\frac{10}{n^{3/2}} = 10\sum_{n=1}^{\infty}\frac{1}{n^{3/2}},$

convergent p-series

72. $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^2+5}$

converges by limit comparison to convergent p-series

$\sum \frac{1}{n^2}$.

73. Diverges by nth-Term Test

$\lim_{n\to\infty} a_n = \infty$

74. Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

75. Convergent geometric series

$\left(r = \frac{7}{8} < 1\right)$

76. Diverges by nth-Term Test

$\lim_{n\to\infty} a_n = \frac{3}{2}$

77. Convergent geometric series $\left(r = 1/\sqrt{e}\right)$ or Integral Test

78. Converges (conditionally) by Alternating Series Test

79. Converges (absolutely) by Alternating Series Test

80. Diverges by comparison to Divergent Harmonic Series:

$\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

81. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

82. $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

(i) $s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$

$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$

Adding: $s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = s_{3n}$

(ii) $\lim_{n\to\infty} s_n = s$ (In fact, $s = \ln 2$.)

$s \ne 0$ because $s > \frac{1}{2}$.

$S = \lim_{n\to\infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$

So, $S \ne s$.

Section 9.6 The Ratio and Root Tests

1. $\displaystyle\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = (n+1)(n)(n-1)$

2. $\displaystyle\frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$

3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid because $1 = \frac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) = \frac{(2n + 2)!}{2^{n+1}(n + 1)!}.$$

To do this, note that:

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) &= \left[1 \cdot 3 \cdot 5 \cdots (2n - 1)\right](2n + 1) \\ &= \frac{(2n)!}{2^n n!} \cdot (2n + 1) \quad \text{(Induction hypothesis)} \\ &= \frac{(2n)!(2n + 1)}{2^n n!} \cdot \frac{(2n + 2)}{2(n + 1)} \\ &= \frac{(2n)!(2n + 1)(2n + 2)}{2^{n+1} n!(n + 1)} \\ &= \frac{(2n + 2)!}{2^{n+1}(n + 1)!} \end{aligned}$$

The formula is valid for all $n \geq 1$.

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid because $\frac{1}{1} = \frac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} = \frac{2^n n!(2n - 3)(2n - 1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} = \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!}.$$

To do this, note that:

$$\begin{aligned} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} \cdot \frac{1}{(2n - 3)} \\ &= \frac{2^n n!\cancel{(2n - 3)}(2n - 1)}{(2n)!} \cdot \frac{1}{\cancel{(2n - 3)}} \\ &= \frac{2^n n!(2n - 1)}{(2n)!} \cdot \frac{(2n + 1)(2n + 2)}{(2n + 1)(2n + 2)} \\ &= \frac{2^n(2)(n + 1)n!(2n - 1)(2n + 1)}{(2n)!(2n + 1)(2n + 2)} \\ &= \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!} \end{aligned}$$

The formula is valid for all $n \geq 3$.

5. $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.875$

Matches (d).

6. $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 = 1.03$

Matches (c).

7. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots$

$S_1 = 9$

Matches (f).

8. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b).

9. $\sum_{n=1}^{\infty}\left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots$

$S_1 = 2, S_2 = 3.31$

Matches (a).

10. $\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

$S_1 = 4$

Matches (e).

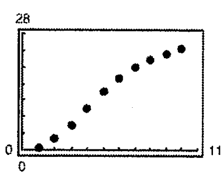
11. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3(1/2)^{n+1}}{n^3(1/2)^n}$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\frac{1}{2} = \frac{1}{2} < 1, \text{ converges}$$

(b)

n	5	10	15	20	25
S_n	13.7813	24.2363	25.8468	25.9897	25.9994

(c)



(d) The sum is approximately 26.

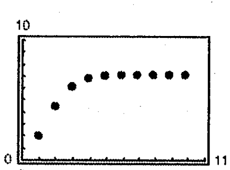
(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of partial sums approaches the sum of the series.

12. (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{(n+1)^2+1}{(n+1)!}}{\dfrac{n^2+1}{n!}} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1$, converges

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548

(c)



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/5^{(n+1)}}{1/5^n}\right| = \lim_{n\to\infty}\frac{5^n}{5^{n+1}} = \frac{1}{5} < 1$$

Therefore, the series converges by the Ratio Test.

14. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!}$$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Therefore, the series converges by the Ratio Test.

15. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

16. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{(n+1)}/(n+1)!}{2^n/n!}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Therefore, the series converges by the Ratio Test.

17. $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(6/5)^{n+1}}{n(6/5)^n}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{6}{5}\right) = \frac{6}{5} > 1$$

Therefore, the series diverges by the Ratio Test.

18. $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(7/8)^{n+1}}{n(7/8)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{7}{8}\right) = \frac{7}{8} < 1$$

Therefore, the series converges by the Ratio Test.

19. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)/4^{n+1}}{n/4^n} = \lim_{n\to\infty}\frac{n+1}{4n} = 1/4 < 1$$

Therefore, the series converges by the Ratio Test.

20. $\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{(n+1)}/(n+1)^4}{5^n/n^4}\right|$$

$$= \lim_{n\to\infty}5\left(\frac{n+1}{n}\right)^4 = 5 > 1$$

Therefore, the series diverges by the Ratio Test.

21. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3/3^{(n+1)}}{n^3/3^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\frac{1}{3} = \frac{1}{3} < 1$$

Therefore, the series converges by the Ratio Test.

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 9.14, the series converges.

Note: The Ratio Test is inconclusive because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

The series converges conditionally.

23. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3/2)^{n+1}}{n^2+2n+1}\cdot\frac{n^2}{(3/2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

25. $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right| = \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

26. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)^5}\cdot\frac{n^5}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

27. $\sum_{n=0}^{\infty} \frac{e^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}/(n+1)!}{e^n/n!}$$

$$= \lim_{n\to\infty} e\left(\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty}\frac{e}{n+1} = 0$$

Therefore, the series converges by the Ratio Test.

28. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

Therefore, the series converges by the Ratio Test.

29. $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{6^{n+1}/(n+2)^{n+1}}{6^n/(n+1)^n} = \lim_{n\to\infty}\frac{6}{n+2}\left(\frac{n+1}{n+2}\right)^n = 0\left(\frac{1}{e}\right) = 0.$$

To find $\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$: Let $y = \left(\frac{n+1}{n+2}\right)^n$

$$\ln y = n\ln\left(\frac{n+1}{n+2}\right) = \frac{\ln(n+1) - \ln(n+2)}{1/n}$$

$$\lim_{n\to\infty}[\ln y] = \lim_{n\to\infty}\left[\frac{1/(n+1) - 1/(n+2)}{-1/n^2}\right] = \lim_{n\to\infty}\left[\frac{-n^2[(n+2)-(n+1)]}{(n+1)(n+2)}\right] = -1$$

by L'Hôpital's Rule. So, $y \to \frac{1}{e}$.

Therefore, the series converges by the Ratio Test.

30. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

31. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5^{n+1}/(2^{n+1}+1)}{5^n/(2^n+1)} = \lim_{n\to\infty}\frac{5(2^n+1)}{(2^{n+1}+1)} = \lim_{n\to\infty}\frac{5(1+1/2^n)}{2+1/2^n} = \frac{5}{2} > 1$$

Therefore, the series diverges by the Ratio Test.

32. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{4n+4}}{(2n+3)!}\cdot\frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty}\frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5\cdots(2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1\cdot 3} - \frac{2!}{1\cdot 3\cdot 5} + \frac{3!}{1\cdot 3\cdot 5\cdot 7} - \cdots$.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2\cdot 4\cdot 6\cdots 2n}{2\cdot 5\cdot 8\cdots(3n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 4\cdots 2n(2n+2)}{2\cdot 5\cdots(3n-1)(3n+2)}\cdot\frac{2\cdot 5\cdots(3n-1)}{2\cdot 4\cdots 2n}\right| = \lim_{n\to\infty}\frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2\cdot 4}{2\cdot 5} - \frac{2\cdot 4\cdot 6}{2\cdot 5\cdot 8} + \cdots$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{1}{5^n}\right]^{1/n} = \frac{1}{5} < 1$$

Therefore, by the Root Test, the series converges.

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\frac{1}{n^n}\right]^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, by the Root Test, the series converges.

37. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \lim_{n\to\infty}\frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

39. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n+2}{n+3}\right)^n}$$

$$= \lim_{n\to\infty}\frac{3n+2}{n+3} = 3 > 1$$

Therefore, the series diverges by the Root Test.

40. $\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{n-2}{5n+1}\right|^n}$$
$$= \lim_{n\to\infty}\left|\frac{n-2}{5n+1}\right| = \frac{1}{5} < 1$$

Therefore, the series converges by the Root Test.

41. $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

42. $\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$$
$$= \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

43. $\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(2\sqrt[n]{n}+1\right)^n} = \lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right)$$

To find $\lim_{n\to\infty}\sqrt[n]{n}$, let $y = \lim_{n\to\infty}\sqrt[n]{n}$. Then

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[n]{n}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0.$$

So, $\ln y = 0$, so $y = e^0 = 1$ and

$\lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right) = 2(1) + 1 = 3.$

Therefore, by the Root Test, the series diverges.

44. $\sum_{n=0}^{\infty}e^{-3n} = \sum_{n=0}^{\infty}\frac{1}{e^{3n}}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{e^{3n}}} = \lim_{n\to\infty}\left(\frac{1}{3^n}\right)^{1/n} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

45. $\sum_{n=1}^{\infty}\frac{n}{3^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n}{3^n}\right)^{1/n} = \lim_{n\to\infty}\frac{n^{1/n}}{3} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

Note: You can use L'Hôpital's Rule to show $\lim_{n\to\infty}n^{1/n} = 1$:

Let $y = n^{1/n}$, $\ln y = \frac{1}{n}\ln n = \frac{\ln n}{n}$

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0 \Rightarrow y \to 1$$

46. $\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{500}\right)^n} = \lim_{n\to\infty}\left(\frac{n}{500}\right) = \infty$$

Therefore, by the Root Test, the series diverges.

47. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n}$$
$$= \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0 - 0 = 0 < 1$$

Therefore, by the Root Test, the series converges.

48. $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty}\frac{\ln n}{n} = 0 < 1$$

Therefore, by the Root Test, the series converges.

49. $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty}\frac{n^{1/n}}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

50. $\sum_{n=1}^{\infty}\frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty}\frac{(n!)^n}{(n^2)^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{(n!)}{(n^2)^n}} = \lim_{n\to\infty}\frac{n!}{n^2} = \infty$$

Therefore, by the Root Test, the series diverges.

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

52. $\sum_{n=1}^{\infty} \frac{100}{n} = 100\sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

53. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is a convergent p-series.

54. $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

Because $|r| = \frac{2\pi}{3} > 1$, this is a divergent Geometric Series.

55. $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

$$\lim_{n\to\infty} \frac{5n}{2n-1} = \frac{5}{2}$$

Therefore, the series diverges by the nth-Term Test

56. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

$$\lim_{n\to\infty} \frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$

Because $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

58. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a Limit Comparison Test with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

59. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

$$\lim_{n\to\infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n\to\infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a Limit Comparison Test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

60. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the nth-Term Test.

61. $\left|\frac{\cos n}{3^n}\right| \le \frac{1}{3^n}$

Therefore the series $\sum_{n=1}^{\infty} \left|\frac{\cos n}{3^n}\right|$ converges by Direct comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. So, $\sum \frac{\cos n}{3^n}$ converges.

62. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

63. $\sum_{n=1}^{\infty} \frac{n!}{n7^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!/(n+1)7^{n+1}}{n!/n7^n}\right|$$
$$= \lim_{n\to\infty}\frac{(n+1)!\,n}{(n+1)\,n!}7$$
$$= \lim_{n\to\infty} 7n = \infty$$

Therefore, the series diverges by the Ratio Test.

64. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

65. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^n}{(n+1)!}\cdot\frac{n!}{3^{n-1}}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges. (Absolutely)

66. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right| = \lim_{n\to\infty}\frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

67. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot5\cdot7\cdots(2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{3\cdot5\cdot7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot5\cdot7\cdots(2n+1)}{(-3)^n}\right| = \lim_{n\to\infty}\frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

68. $\sum_{n=1}^{\infty} \frac{3\cdot5\cdot7\cdots(2n+1)}{18^n(2n-1)n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!}\cdot\frac{18^n(2n-1)n!}{3\cdot5\cdot7\cdots(2n+1)}\right| = \lim_{n\to\infty}\frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{1}{18}$$

Therefore, by the Ratio Test, the series converge.

69. (a) and (c) are the same.

$$\sum_{n=1}^{\infty}\frac{n5^n}{n!} = \sum_{n=0}^{\infty}\frac{(n+1)5^{n+1}}{(n+1)!}$$
$$= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots$$

70. (b) and (c) are the same.

$$\sum_{n=0}^{\infty}(n+1)\left(\tfrac{3}{4}\right)^n = \sum_{n=1}^{\infty}n\left(\tfrac{3}{4}\right)^{n-1}$$
$$= 1 + 2\left(\tfrac{3}{4}\right) + 3\left(\tfrac{3}{4}\right)^2 + 4\left(\tfrac{3}{4}\right)^3 + \cdots$$

71. (a) and (b) are the same.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$
$$= 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots$$

72. (a) and (b) are the same.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)2^{n-1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}$$
$$= \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \cdots$$

73. Replace n with $n + 1$.

$$\sum_{n=1}^{\infty}\frac{n}{7^n} = \sum_{n=0}^{\infty}\frac{n+1}{7^{n+1}}$$

74. Replace n with $n + 2$.

$$\sum_{n=2}^{\infty}\frac{9^n}{(n-2)!} = \sum_{n=0}^{\infty}\frac{9^{n+2}}{n!}$$

75. (a) Because

$$\frac{3^{10}}{2^{10}10!} \approx 1.59\times 10^{-5},$$

use 9 terms.

(b) $\displaystyle\sum_{k=1}^{9}\frac{(-3)^k}{2^k k!} \approx -0.7769$

76. (a) Use 10 terms, $k = 9$, see Exercise 3.

(b) $$\sum_{k=0}^{\infty}\frac{(-3)^k}{1\cdot 3\cdot 5\ldots(2k+1)} = \sum_{k=0}^{\infty}\frac{(-3)^k 2^k k!}{(2k)!(2k+1)}$$
$$= \sum_{k=0}^{\infty}\frac{(-6)^k k!}{(2k+1)!} \approx 0.40967$$

77. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4n-1)/(3n+2)a_n}{a_n}\right|$$
$$= \lim_{n\to\infty}\frac{4n-1}{3n+2} = \frac{4}{3} > 1$$

The series diverges by the Ratio Test.

78. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)/(5n-4)a_n}{a_n}\right|$$
$$= \lim_{n\to\infty}\frac{2n+1}{5n-4} = \frac{2}{5} < 1$$

The series converges by the Ratio Test.

79. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\sin n+1)/(\sqrt{n})a_n}{a_n}\right|$$
$$= \lim_{n\to\infty}\frac{\sin n+1}{\sqrt{n}} = 0 < 1$$

The series converges by the Ratio Test.

80. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\cos n+1)/(n)a_n}{a_n}\right|$$
$$= \lim_{n\to\infty}\frac{\cos n+1}{n} = 0 < 1$$

The series converges by the Ratio Test.

81. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(1+(1)/(n))a_n}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

The Ratio Test is inconclusive.

But, $\lim_{n\to\infty} a_n \neq 0$, so the series diverges.

82. The series diverges because $\lim_{n\to\infty} a_n \neq 0$.

$$a_1 = \tfrac{1}{4}$$
$$a_2 = \left(\tfrac{1}{4}\right)^{1/2} = \tfrac{1}{2}$$
$$a_3 = \left(\tfrac{1}{2}\right)^{1/3} \approx 0.7937$$

In general, $a_{n+1} > a_n > 0$.

83. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 2\cdots n(n+1)}{1\cdot 3\cdots(2n-1)(2n+1)}}{\dfrac{1\cdot 2\cdots n}{1\cdot 3\cdots(2n-1)}}\right|$$
$$= \lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{1}{2} < 1$$

The series converges by the Ratio Test.

84. $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{3}$$

Let $\quad y = \lim_{n\to\infty}\sqrt[n]{n+1}$

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[n]{n+1}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\ln(n+1)$$
$$= \lim_{n\to\infty}\frac{\ln(n+1)}{n} = \frac{1}{n+1} = 0.$$

Because $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty}\frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

85. $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

86. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{1\cdot 2\cdot 3\cdots(2n-1)(2n)(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(2n-1)}}\right|$$

$$= \lim_{n\to\infty} \frac{2n+1}{(2n)(2n+1)} = 0 < 1$$

The series converges by the Ratio Test.

87. $\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(x/3)^{n+1}}{2(x/3)^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

For the series to converge, $\left|\frac{x}{3}\right| < 1 \Rightarrow -3 < x < 3$.

For $x = 3$, $\sum_{n=0}^{\infty} 2(1)^n$ diverges.

For $x = -3$, $\sum_{n=0}^{\infty} 2(-1)^n$ diverges.

88. $\sum_{n=0}^{\infty} \left(\frac{x-3}{5}\right)^n$, Geometric series

For the series to converge,

$$\left|\frac{x-3}{5}\right| < 1 \Rightarrow |x-3| < 5$$
$$\Rightarrow -2 < x < 8.$$

For $x = 8$, $\sum_{n=0}^{\infty} 1^n$ diverges.

For $x = -2$, $\sum_{n=0}^{\infty} (-1)^n$ diverges.

(Note: You could also use the Ratio Test.)

89. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}/(n+1)}{x^n/n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n}{n+1}(x+1)\right| = |x+1|$$

For the series to converge,

$$|x+1| < 1 \Rightarrow -1 < x+1 < 1$$
$$\Rightarrow -2 < x < 0.$$

For $x = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

For $x = -2$, $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

90. $\sum_{n=0}^{\infty} 3(x-4)^n$, Geometric series

For the series to converge,

$$|x-4| < 1 \Rightarrow -1 < x-4 < 1 \Rightarrow 3 < x < 5.$$

For $x = 1$, $\sum_{n=0}^{\infty} 3(-3)^n$ diverges.

For $x = -1$, $\sum_{n=0}^{\infty} 3(-5)^n$ diverges.

91. $\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\left|\frac{x}{2}\right|^{n+1}}{n!\left|\frac{x}{2}\right|^n}$$
$$= \lim_{n\to\infty} (n+1)\left|\frac{x}{2}\right| = \infty$$

The series converges only at $x = 0$.

92. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+1|^{n+1}}{(n+1)!} \bigg/ \frac{|x+1|^n}{n!} = \lim_{n\to\infty} \frac{|x+1|}{n+1} = 0$$

The series converges for all x.

93. See Theorem 9.17, page 627.

94. See Theorem 9.18, page 630.

95. No. Let $a_n = \dfrac{1}{n + 10{,}000}$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.

96. (a) Converges (Ratio Test)
(b) Inconclusive (See Ratio Test)
(c) Diverges (Ratio Test)
(d) Diverges (Root Test)
(e) Inconclusive (See Root Test)
(f) Diverges (Root Test, $e > 1$)

97. The series converges absolutely. See Theorem 9.17.

98. For $0 < a_n < 1$, $a_n < \sqrt{a_n}$.

Thus, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is the lower series, indicated by the round dots.

99. Assume that

$\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$|a_{n+1}| > |a_n|,\ n > N \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges.

100. First, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \le r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. So, for $n > N\ |a_n| < R^n$ and because the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, you can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for infinitely many $n > M$. So, for infinitely many $n > M$, you have $|a_n| > R^n > 1$ which implies that $\lim_{n\to\infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

101. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{3/2} = 1$$

102. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1}\right|$$
$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1/2} = 1$$

103. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^4} \cdot \frac{n^4}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^4 = 1$$

104. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^p} \cdot \frac{n^p}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

105. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, p-series

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{n^{p/n}} = 1$$

So, the Root Test is inconclusive.

Note: $\lim_{n\to\infty} n^{p/n} = 1$ because if $y = n^{p/n}$, then

$\ln y = \frac{p}{n} \ln n$ and $\frac{p}{n} \ln n \to 0$ as $n \to \infty$.

So $y \to 1$ as $n \to \infty$.

106. Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p} = 1, \text{ inconclusive.}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n(\ln n)^p}} = \lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}}$$

$\lim_{n\to\infty} n^{1/n} = 1$. Furthermore, let $y = (\ln n)^{p/n} \Rightarrow$

$\ln y = \frac{p}{n} \ln(\ln n)$.

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{p \ln(\ln n)}{n} = \lim_{n\to\infty} \frac{p}{\ln(n)(1/n)} = 0 \Rightarrow \lim_{n\to\infty} (\ln n)^{p/n} = 1.$$

So, $\lim_{n\to\infty} \frac{1}{n^{1/n}(\ln n)^{p/n}} = 1$, inconclusive.

107. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$, x positive integer

(a) $x = 1$: $\sum \frac{(n!)^2}{n!} = \sum n!$, diverges

(b) $x = 2$: $\sum \frac{(n!)^2}{(2n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(2n+2)!} \Big/ \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

(c) $x = 3$: $\sum \frac{(n!)^2}{(3n)!}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(3n+3)!} \Big/ \frac{(n!)^2}{(3n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

(d) Use the Ratio Test:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{[x(n+1)]!} \Big/ \frac{(n!)^2}{(xn)!} = \lim_{n\to\infty} (n+1)^2 \frac{(xn)!}{(xn+x)!}$$

The cases $x = 1, 2, 3$ were solved above. For $x > 3$, the limit is 0. So, the series converges for all integers $x \geq 2$.

108. For $n = 1, 2, 3, \ldots, -|a_n| \le a_n \le |a_n| \Rightarrow -\sum_{n=1}^{k}|a_n| \le \sum_{n=1}^{k} a_n \le \sum_{n=1}^{k}|a_n|$.

Taking limits as $k \to \infty, -\sum_{n=1}^{\infty}|a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}|a_n| \Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}|a_n|$.

109. First prove Abel's Summation Theorem:

If the partial sums of $\sum a_n$ are bounded and if $\{b_n\}$ decreases to zero, then $\sum a_n b_n$ converges.

Let $S_k = \sum_{i=1}^{k} a_i$. Let M be a bound for $\{|S_k|\}$.

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &= S_1b_1 + (S_2 - S_1)b_2 + \cdots + (S_n - S_{n-1})b_n \\ &= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_nb_n \\ &= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_nb_n \end{aligned}$$

The series $\sum_{i=1}^{\infty} S_i(b_i - b_{i+1})$ is absolutely convergent because $|S_i(b_i - b_{i+1})| \le M(b_i - b_{i+1})$ and $\sum_{i=1}^{\infty}(b_i - b_{i+1})$ converges to b_1.

Also, $\lim_{n\to\infty} S_nb_n = 0$ because $\{S_n\}$ bounded and $b_n \to 0$. Thus, $\sum_{n=1}^{\infty} a_nb_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_ib_i$ converges.

Now let $b_n = \frac{1}{n}$ to finish the problem.

110. Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n \Big/ \frac{(n-1)!}{n^{n-1}}\left(\frac{19}{7}\right)^{n-1}\right] = \lim_{n\to\infty}\left[\frac{n \cdot n^{n-1}}{(n+1)^n}\left(\frac{19}{7}\right)\right] = \lim_{n\to\infty}\left[\frac{1}{(1+(1/n))^n}\left(\frac{19}{7}\right)\right] = \frac{19}{7}\cdot\frac{1}{e} < 1$$

So, the series converges.

Section 9.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

2. $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

3. $y = e^{-1/2}[(x+1) + 1]$

Linear

Matches (a)

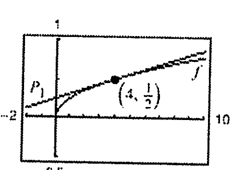
4. $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

Cubic

Matches (b)

5. $f(x) = \frac{\sqrt{x}}{4}, C = 4, f(4) = \frac{1}{2}$

$$\begin{aligned} f'(x) &= \frac{1}{8\sqrt{x}}, f'(4) = \frac{1}{16} \\ P_1(x) &= f(4) + f'(4)(x - 4) \\ &= \frac{1}{2} + \frac{1}{16}(x - 4) \\ &= \frac{1}{16}x + \frac{1}{4} \end{aligned}$$



P_1 is the first-degree Taylor polynomial for f at 4.