BC Calculus 9.7-9.10 Review WS #3

1) Find the radius and interval of convergence for

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$

(b) $\sum_{n=0}^{\infty} (2n)!(x-5)^n$

2) Let $f(x)=\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ for all x for which the series converges.

(a) Find the interval of convergence of this series.

(b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.

(c) Estimate the error involved in the approximation in part (b). Show your reasoning.

3) Let f be the function given by $f(x) = \cos\left(3x + \frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right)\right| < \frac{1}{3000}$.

4) The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by

$$f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)} \text{ and } f(5) = \frac{1}{2}.$$

Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

5) Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x = 1$. If the maximum value of the fifth derivative between $x = 1$ and $x = 3$ is 0.01, that is, $\left|f^{(5)}(x)\right| < 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

6) What are all the values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x < 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$ (E) All real x

7) The coefficient of x^6 in the Taylor series expansion about $x = 0$ for $f(x) = \sin(x^2)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

8) If f is a function such that $f'(x) = \sin(x^2)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x = 0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

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Key

1) Find the radius and interval of convergence for

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2} = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1}(n+1)^2} \cdot \frac{3^n n^2}{(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-2)}{3} \cdot \frac{n^2}{(n+1)^2} \right| = \left| \frac{x-2}{3} \right| < 1 \quad |x-2|<3$

$-1<x<5$

radius $=3$, $c=2$ $\quad [(-1,5)]$

test end pts

test $x=-1$, $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n \cdot n^2} = \frac{+1}{n^2}$ (converging p-series)

test $x=5$ $\sum_{n=0}^{\infty} \frac{(-1)^n(3)^n}{3^n \cdot n^2} = \frac{(-1)^n}{n^2}$ (Alt. series test converging series) $\boxed{I.O.C. = [-1,5]}$

(b) $\sum_{n=0}^{\infty} (2n)!(x-5)^n$

$\lim_{n\to\infty} \left| \frac{(2n+2)!(x-5)^{n+1}}{(2n)!(x-5)^n} \right| =$

$\lim_{n\to\infty} |(2n+2)(2n+1)(x-5)| = \infty$

$\boxed{\text{series converge only at center } c=5}$

2) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ for all x for which the series converges.

(a) Find the interval of convergence of this series.

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{2} \right| < 1 \quad -2<x<2 \quad [(-2,2)]$

test $x=-2$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = (-1)^n$ diverges (n^{th} term)

test $x=2$ $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = 1^n$ diverges (n^{th} term)

$\boxed{I.O.C : (-2,2)}$

(b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.

$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{4}$

$f(-1/2) = 1 - 1/4 + \frac{1}{16} = 0.8125 = \boxed{\frac{13}{16}}$

(c) Estimate the error involved in the approximation in part (b). Show your reasoning.

for $x=-1/2$, this is a converging alternate series.

Error is magnitude of 1^{st} unused term.

$\text{Error} \le \left| \frac{-1}{64} \right| = \boxed{\frac{1}{64}}$

3) Let f be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x=0$.

$\frac{f^n(c)}{n!}(x-c)^n$

(a) Find $P(x)$.

$f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ $\quad f(0)=\frac{\sqrt{3}}{2}$

$f'(x)=-3\sin(3x+\pi/6)$, $f'(0)=\frac{-3}{2}$

$f''(x)=-9\cos(3x+\pi/6)$ $\quad f''(0)=\frac{-9\sqrt{3}}{2}$

$f'''(x)=27\sin(3x+\pi/6)$ $\quad f'''(0)=\frac{27}{2}$

$f^{(4)}(x)=81\cos(3x+\pi/6)$ $\quad f^4(0)=\frac{81\sqrt{3}}{2}$

$f^{(5)}(x)=-243\sin(3x+\pi/6)$ $\quad f^5(0)=\frac{-243}{2}$

$f(x)\approx P_4(x)=$

$$\boxed{\frac{\sqrt{3}}{2}-\frac{3}{2}x-\frac{9\sqrt{3}}{4}x^2+\frac{9}{4}x^3+\frac{27\sqrt{3}}{16}x^4}$$

$$P_4(x)=\frac{\sqrt{3}}{2}-\frac{3}{2}(x-0)-\frac{\frac{9\sqrt{3}}{2}}{2!}(x-0)^2+\frac{\frac{27}{2}}{3!}(x-0)^3+\frac{\frac{81\sqrt{3}}{2}}{4!}(x-0)^4$$

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

$$R_n(x)=\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

$x=1/6$

$c=0$

$*\ f^5(x)=-243\sin(3x+\pi/6)$

Max value of $|f^5(z)|=243$

$$=\left|\frac{f^5(z)}{5!}\left(\frac{1}{6}-0\right)^5\right|\leq\left|\frac{243}{5!}\left(\frac{1}{6}\right)^5\right|=\boxed{0.0002604<\frac{1}{3000}}$$

4) The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by

$$f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)} \text{ and } f(5) = \frac{1}{2}.$$

X

Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

$x = 6$
$c = 5$

$$R_6(6) = \left| \frac{f^7(z)}{7!}(6-5)^7 \right| = \left| \frac{\frac{-7!}{2^7(9)}}{7!}(1)^7 \right| = \frac{1}{1152} < \frac{1}{1000}$$

5) Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x = 1$. If the maximum value of the fifth derivative between $x = 1$ and $x = 3$ is 0.01, that is, $\left|f^{(5)}(x)\right| < 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

$c = 1, \; x = 3$

$$R_4(3) = \left| \frac{f^5(z)}{5!}(3-1)^5 \right| \le \left| \frac{0.01}{5!}(2)^5 \right| = \boxed{0.0026666}$$

6) What are all the values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x < 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$ $\boxed{\text{(E) All real } x}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1 \qquad \text{I.O.C}: (-\infty, \infty)$$

~~Test $\cancel{x=-1}$ $\cancel{\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}}$ converges by AST~~ | ~~Test $\cancel{x=1}$ $\cancel{\sum_{n=1}^{\infty} \frac{1^n}{n!}}$ converges by Direct Comparison~~

7) The coefficient of x^6 in the Taylor series expansion about $x=0$ for $f(x) = \sin(x^2)$ is

$\boxed{\text{(A) } -\frac{1}{6}}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(x^2) = x^2 \boxed{- \frac{x^6}{3!}} + \frac{x^{10}}{5!}$$

$\hookrightarrow$ coefficient is $\frac{-1}{3!}$ or $\boxed{\frac{-1}{6}}$

8) If f is a function such that $f'(x) = \sin(x^2)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x=0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 $\boxed{\text{(D) } -\frac{1}{42}}$ (E) $-\frac{1}{7!}$

$$f'(x) = \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$$

$$\int f'(x)\,dx = C + \frac{x^3}{3} \boxed{- \frac{x^7}{3! \cdot 7}} + \frac{x^{11}}{5! \cdot 11}$$

$\hookrightarrow$ coefficient is $\frac{-1}{3!(7)}$ or $\boxed{\frac{-1}{42}}$